RADON - NIKODÝM PROPERTY

in

LOCALLY CONVEX SPACES

By

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To my dear wife Haibat and loving sons Ivan and Waseem without whose support this never would have been written.

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ABSTRACT

Our object in this thesis is to study the Radon-Nikodym property (RNP) in the class of locally convex spaces (ℓ.c.s). We divide this work into four chapters. Chapter one is an introductory chapter, containing various definitions, theorems and notations which are needed in the other chapters, such as vector measure, RNP, bounded variation of vector measures, dentability, etc.

Chapter 2

The Liapounoff convexity theorem and the Uhl generalization of this theorem on the class of Banach spaces which are either reflexive or separable dual spaces are given in chapter two. We give two examples due to Uhl to show that this generalization cannot be improved under the current hypotheses.

Our goal in chapter two is to give a generalization of the Uhl convexity theorem on the range of vector measures in the class of locally convex spaces with a Radon-Nikodym derivative.

Chapter 3

Rieffel proved the fundamental Radon-Nikodým Theorem (RNT) for Banach spaces. Since then, various efforts have been made to extend Rieffels (RNT) to locally convex spaces.

Saab extended Rieffels Theorem in the class of quasi-complete locally convex spaces with property that every bounded set is metrizable.

Our goal is to give a generalization of the result of Saab using the same technique he used for general locally convex spaces and this is contained in chapter three.

Chapter 4

The equivalence between the Radon-Nikodym property and Bishop-Phelps property (BPP) in the class of Banach spaces was proved by J. Bourgin . We prove the relation between these properties in the class of locally convex spaces with the property that every bounded set is metrizable. To prove this we use a new definition of (BPP) in locally convex spaces and the result of Saab.

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CHAPTER I

This chapter contains the definitions notations and theorems necessary as background information for the following chapters.

(I) A topological vector space E over R will be called a "locally convex space" (ℓ.c.s) if it is a Hausdorff space such that every neighbourhood (nhd) of x in E contains a convex nhd of x. Equivalently E is said to be ℓ.c.s if the convex nhd's of 0 form a base at 0 with intersection $\{0\}$.

Let $\{P_\alpha\}_{\alpha \varepsilon A}$ be an arbitrary family of semi-norms determining the topology τ on E.

Notation 1.1

The letter B will always be used to denote a Banach space with $||\cdot||$ as norm.

The letter E will always be used to denote a ℓ.c.s with $\{P_\alpha\}_{\alpha\varepsilon A}$ as a family of semi-norms that make the topology τ on E.

The letter L will denote "topological vector space" (TVS) in general.

The triple $(X, \sum, \mu)$ will be used to denote a probability measure space: X is a set, $\sum$ is a σ-algebra of subset of X, and μ is a probability measure on $\sum$.

$\sum^+$ denotes the family

$$\{A \varepsilon \textstyle\sum : \mu(A) > 0\} .$$

m will denote a countably additive vector measure on $\sum$, with a value either in B or in E.

Definition 1.2

For every semi-norm p the "p-variation" of a vector measure $m : \sum \to L$ over a measurable set $A \in \sum$ is defined by:

$$|m|_p(A) = \sup_{A_i \in \pi} \{ \sum_{i=1}^{n} p(m(A_i)) \}$$ where π is the set of all finite partitions of A by means of measurable sets. For a Banach space B, we will denote the variation of m over $A \in \sum$ by :

$$|m|(A) = \sup_{A_i \in \pi} \{ \sum_{i=1}^{n} \|m(A_i)\| \}.$$

The vector measure m is said to have a finite variation if $|m|_p(X) < \infty$.

A set $A \in \sum$ is an atom of m if $m(A) \neq 0$ and if $C \in \Sigma$ and $C \subseteq A$ imply $m(C) = 0$ or $m(C) = m(A)$. If for each $A \in \Sigma$ A is not an atom of m, then the vector measure m is said to be non-atomic.

A subset of L is called conditionally compact if its closure is compact in its relative topology.

A subset A of L is called totally bounded if for each nhd V of 0 in L there is a finite subset $A_o \subseteq A$ such that $A \subseteq A_o + V$.

A subset A of L is said to be pre-compact if and only if the closure of A in the completion of L, (denoted by $\tilde{L}$) is compact.

Equivalently a subset of E is pre-compact if and only if it is totally bounded (See Schaefer [44]).

A topological vector space L is called a quasi-complete TVS if every closed bounded subset of L is complete.

In a Hausdorff quasi-complete TVS every pre-compact subset is relatively/conditionally compact (see Schaefer [44]).

A subset A of a TVS L is said to be a "barrel" in L if A is an absorbing, convex, balanced and closed subset of L. We say L is barrelled if and only if each barrel is a nhd of 0.

A quasi-complete ℓ.c.s E is said to have the "BM" property if and only if every closed bounded subset of E is metrizable This is an extension of the definition of the BM property originally given by Chi [9] although this has not been referred to by Saab [43]. Chi's definition stated: For every bounded subset $A \subset \ell^1_N(E)$, the space of absolutely summable sequences, there exists an absolutely convex closed bounded and metrizable subset $M \subseteq E$ such that $\sum_{i=1}^{\infty} p_M(x_i) < 1$, for every $(x_i) \in A$.

The two definitions of the BM property are not equivalent.

Let $(BM)^*$ denote the BM property given by Chi and the BM be that of Saab.

The following two properties together imply the $(BM)^*$ property.

(a) Every absolutely convex closed, bounded subset of E is metrizable.

(b) For every bounded subset $A \subseteq \ell^1_N(E)$, there exists an absolutely convex closed bounded subset $M \subseteq E$ such that

$$\sum_{i=1}^{\infty} p_M(x_i) < 1, \quad \text{for all } x_i \in A.$$

See Chi [9].

However, the BM property of Saab is equivalent to the property (a). To see this, we first show that (a) implies the BM property.

For every bounded subset C of E, the closed absolutely convex hull of C is bounded and so is metrizable by virtue of the property (a). Therefore, C is metrizable. To próve the converse, let C be an absolutely closed bounded subset of E. Since C is bounded, it is metrizable by virtue of the BM property.

(II) Radon and Nikodým's theorems contain a condition whereby one measure varies smoothly with respect to another. We begin by stating one version of the Radon-Nikodým theorem (RNT) which, in spite of its restrictive hypotheses, contains all the important features needed for subsequent generalization.

Theorem 1.3 (Radon-Nikodým theorem on R)

Let (X, Σ, μ) be a probability space and $m : \Sigma \to R$ be a measure which is absolutely continuous with respect to $\mu(m << \mu)$ and is of finite variation $(|m|(X) < \infty)$. Then there exists a μ-integrable function $f : X \to R$ such that:

$$\int_A f\,d\mu = m(A) \quad \text{for each} \quad A \in \Sigma .$$

If R in both instances above is replaced by a general Banach space B or $\ell.c.s$ E the resulting formal statement requires interpretation. (i.e. what is an absolutely continuous vector measure and what does $\int_A f\,d\mu$ mean ?).

Definition 1.4

A vector measure $m : X \to E$ is said to be absolutely continuous with respect to μ $(m << \mu)$ if for each $A \in \Sigma$ the condition $\mu(A) = 0$ implies that $m(A) = 0$.

Definition 1.5

Let (X, Σ, μ) be a probability measure. A vector measure m on Σ which is absolutely continuous with respect to μ has an average range defined by:

$$AR(m) = \left\{ \frac{m(A)}{\mu(A)} : A \in \Sigma,\ \mu(A) > 0 \right\}.$$

A Banach-valued integral as defined by Dunford (1937) allows quite general functions to be integrated, but yields a very weak theory of integration which is often of limited value. The Pettis integral is somewhat stronger.

We will be looking at the Bochner integral in our work. It is defined in such a way that the simple functions are L^1 - dense in the space of Bochner integrable functions.

Because of this density, many of the theorems for the Bochner integral closely parallel their real counterparts.

Let (X, Σ, μ) be a complete probability space. A function $S : X \to B$ is a simple function if S is represented in the form

$$\sum_{i=1}^{n} x_i \chi_{A_i}$$

for a distinct $x_i \in B$, $i = 1,2,\ldots, n$ with a finite partition $\{A_i\}_{i=1}^{n}$ of X chosen from Σ.

For any $A \in \Sigma$ and a simple function S as above, define the Bochner integral of S over A by :-

$$\int_A S\, d\mu = \int_A \sum_{i=1}^{n} x_i \chi_{A_i}\, d\mu = \sum_{i=1}^{n} x_i\, \mu\, (A \cap A_i).$$

Generally, a function $f : X \to B$ is said to be a Bochner integrable, (written $f \in L^1_B(X, \Sigma, \mu)$ or sometimes $f \in L^1_B(\mu)$ when no confusion arises), if there exists a sequence of simple functions $\{S_n\}_{n=1}^{\infty}$ such that

(a) $\lim\limits_n S_n(x) = f(x) \quad \mu$.a.e.

(b) $\lim\limits_n \int_X \| f(x) - S_n(x) \| \, d\mu = 0.$

When f is Bochner integrable and $A \in \Sigma$ the $\lim\limits_n \int_A S_n \, d\mu$ exists (in the norm sense) and is independent of the particular choice of a sequence of simple functions satisfying (a) and (b). Therefore, we define $\int_A f \, d\mu = \lim\limits_n \int_A S_n \, d\mu$ for any appropriate choice of S_n's.

Before summarizing those properties of the Bochner integral, we will define the terminology needed under "Definition 1.6" below.

Definition 1.6

Let $f_i : X \to B$ for $i = 1,2,3,4$. If f_1 is the "almost everywhere limit" of a sequence of simple functions, then it is said to be strongly measurable. If $F \circ f_2 : X \to R$ is measurable for any $F \in B^*$ then f_2 is weakly measurable. Suppose that for some $A \in \Sigma$ both $\mu(A) = 0$ and $f_3(X \setminus A)$ is a norm separable set, then f_3 is almost separably valued. Finally if $f_4^{-1}(V) \in \Sigma$ for each norm open set V in B, then f_4 is Borel measurable.

The nexttheorem,(which was proved by Pettis and Bochner), provides useful criteria for determining whether or not a Banach-valued function on a complete probability space is Bochner integrable.

Theorem 1.7

If (X, Σ, μ) is a complete probability space, the following are equivalent:

(a) $f \in L_B^1(\mu)$

(b) f is almost separably valued, weakly measurable and $\int_X \|f(x)\| \, d\mu < \infty$.

(c) f is strongly measurable and $\int_X \|f(x)\| \, d\mu < \infty$

(d) f is almost separably valued, Borel measurable and $\int_X \|f(x)\| \, d\mu < \infty$.

Theorem 1.8

Let $f : X \to B$ be Bochner integrable and $A \in \Sigma$. Then :

(a) $\|\int_A f \, d\mu\| \leq \int_A \|f(x)\| \, d\mu$.

(b) $L_B^1(\mu)$ is a Banach space when equipped with $\|f\|_1 = \int_X \|f(x)\| \, d\mu$.

The simple functions form a dense subset of $L_B^1(\mu)$.

(c) Dominated convergence: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L_B^1(\mu)$ and that $\|f_n(x)\| \leq g(x)$ μ.a.e. for each n where g is μ-integrable.

If $\lim f_n(x) = f(x)$ exists μ.a.e. in the norm sense then:

(i) $f \in L_B^1(\mu)$

(ii) $\lim_n \|(f_n - f)\|_1 = 0$.

(iii) $\int_X f\,d\mu = \lim_n \int_X f_n\,d\mu$.

(d) $\int_A f\,d\mu$ is linear in f and countably additive in A.

(e) If $F \in B^*$ then $F(\int_A f\,d\mu) = \int_A (F \circ f)\,d\mu$.

More generally, if Y is a Banach space and $T : B \to Y$ is a bounded linear operator then:-

$$T \circ f \in L^1_Y(\mu) \text{ and } T(\int_A f\,d\mu) = \int_A T \circ f\,d\mu.$$

(f) If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^1_B(\mu)$ for which $\lim_n \|f_n - f\|_1 = 0$, then there exists a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\lim_i f_{n_i}(x) = f(x)$ μ.a.e.

(III) We will begin with one of the common definitions of the Radon-Nikodým property (RNP).

Definition 1.9

If K is a closed, bounded and convex set in a Banach space B, then K has the RNP for (X, Σ, μ), if for each B-valued measure m on Σ, (which is absolutely continuous with respect to μ and whose average range $AR(m)$ is contained in K), there exists an $f \in L^1_B(\mu)$ such that $m(A) = \int_A f\,d\mu$, for each $A \in \Sigma$. The set K is said to have the RNP if K has the RNP for each probability space (X, Σ, μ).

Finally, let C be a closed convex possibly unbounded subset of B (i.e. C might be B). Then C has the RNP if each of its closed, bounded subsets has the RNP.

Example 1.10

Let us assume that:-

$(X, \sum, \mu) = ([0,1]$, Lebesgue measurable sets, Lebesgue measure $(\lambda))$,

$$B = L^1_{[0,1]}(\lambda),$$

$m : \sum \to B$ defined by $m(A) = \chi_A$ for each $A \varepsilon \sum$. Clearly, $m << \mu$, and $AR(m)$ are subsets of the closed unit ball of B.

Our aim at this stage is to prove that there is no function $f \varepsilon L_B(\lambda)$ such that $m(A) = \int_A f d\lambda$ for all $A \varepsilon \sum$, whence $L^1_{[0,1]}(\lambda)$ does not have the RNP.

Assume temporarily that such a function exists, randomly select any $A \varepsilon \sum$, and any $g \varepsilon L^{\infty}_{[0,1]}(\lambda) = L^{1*}_{[0,1]}$.

The notation (g,h) denotes the value of $g \varepsilon B^*$ at $h \varepsilon B$. We have

$$\int_A (g, f(t))\, dt = (g, \int_A f(t)\, dt) = (g, \chi_A) = \int_A g(t)\, dt.$$

Hence $(g,f(t)) = g(t)$ for all $t \varepsilon ([0,1] \setminus A(g))$ for some $A(g) \varepsilon \sum$, (possibly depending on g) with $\lambda(A(g)) = 0$.

We will use $\{I_n\}_{n=1}^{\infty}$ to denote a listing of all sub-intervals of $[0,1]$ with rational endpoints.

For each n let $g_n = \chi_{I_n} \varepsilon L^{\infty}_{[0,1]}(\lambda)$. Define $A = \bigcup_{n=1}^{\infty} A(g_n)$ and let $x \varepsilon [0,1] \setminus A$. Then

$$\int_{I_n} f(x)(s)\, ds = \int g_n(s)\, f(x)(s)\, ds = g_n(x) = 0 \quad \text{whenever}$$

$x \notin I_n$. This implies that $f(x)(s) = 0$ for almost all $s \in [0,1]$ as long as $x \in [0,1]\setminus A$, where $f : [0,1] \to B$ vanishes λ-a. e. Since this is in contradiction to the hypothesis that $\int_A f\,d\lambda = \chi_A \neq 0$, whenever $\lambda(A) > 0$, so the proof that $L^1_{[0,1]}(\lambda)$ lacks the RNP is complete.

A point $a \in K$ is an extreme point of K if and only if $a = x_1 = x_2$ for each $x_1, x_2 \in K$ and $a = (x_1 + x_2)/2$. The set of extreme points of K is denoted by $ex(K)$.

Proposition 1.11

For any vector space V and for any convex subset $K \subseteq V$ the following are equivalent:

(a) a is an extreme point of K.

(b) If $x_1, x_2 \in K$, $x_1 \neq x_2$, $0 \leq \lambda \leq 1$ and

$$a = \lambda x_1 + (1-\lambda)x_2 \text{ then } \lambda = 0 \text{ or } \lambda = 1$$

(c) If $x_1, x_2 \in K$, $0 < \lambda < 1$, and $a = \lambda x_1 + (1-\lambda)x_2$ then $x_1 = x_2 = a$

(d) $X\setminus\{a\}$ is convex.

Notation 1.12

The following geometric notations will be used:

Let $D \subseteq B$ then

(a) $Co(D)$ is the convex hull of D

(b) $\overline{Co}(D)$ is the norm-closure of the convex hull of D.

(c) $\sigma - Co(D) = \{\sum_{i=1}^{\infty} \alpha_i x_i : x_i \in D,\ \alpha_i \geqslant 0,\ \sum_{i=1}^{\infty} \alpha_i = 1\}$.

(d) $aCo(D)$ is the absolutely convex hull of D.

(e) $\overline{aCo}(D)$ is the norm-closure of the $aCo(D)$.

(f) $U_\varepsilon(x) = \{y \in B : \|y - x\| < \varepsilon\}$.

(g) $U_\varepsilon[x] = \{y \in B : \|y - x\| \leq \varepsilon\}$

(h) $U_\varepsilon(x) = U(0)$ if $\varepsilon = 1$ and $x = 0$

If $D \subseteq R^n$ is compact, the result of Caratheodory (1907) asserts that each point of $\overline{Co}(D)$ is in fact a convex combination of at most $n+1$ points of D. Therefore $Co(D) = \overline{Co}(D)$.

Suppose that K is a compact convex subset of R^n, $x \in ex(K)$ and $\varepsilon > 0$.

Let $D = K \setminus U_\varepsilon(x)$. Since x is extreme, $x \notin Co(D)$ and thus $x \notin \overline{Co}(D)$ by the Caratheodory result above.

In general, if $D \subseteq B$ is closed, and bounded, then

$$Co(D) \subseteq \sigma\text{-}Co(D) \subseteq \overline{Co}(D).$$

and each of these containments may be strict.

<u>Definition</u> 1.13

Let D be a bounded subset of B. Then D is σ-dentable if for each $\varepsilon > 0$ there exists a point $x_\varepsilon \in D$ such that $x_\varepsilon \notin \sigma\text{-}Co(D \setminus U_\varepsilon(x_\varepsilon))$.

D is dentable (respectively c-dentable) if for each $\varepsilon > 0$ there exists a point $x_\varepsilon \in D$ such that $x_\varepsilon \notin \overline{Co}(D \setminus U_\varepsilon(x_\varepsilon))$ (respectively $x_\varepsilon \notin Co(D \setminus U_\varepsilon(x_\varepsilon))$.

Let C be a possibly unbounded subset of B. Then C is

(a) subset σ-dentable if each bounded non-empty subset of C is σ-dentable.

(b) subset dentable (subset c-dentable) if each of its bounded non-empty subsets is dentable (c-dentable).

Example 1.14

The following example illustrates some of the above notations.

Let $B = C[0,1]$, (the Banach space of continuous functions on $[0,1]$ with

$$\| f \| = \max \{|f(t)|: t \in [0,1]\}).$$

Let K be its closed unit ball, $U_1[0]$. Observe that K has exactly two extreme points, namely, the functions identically $+1$ and identically -1. Thus K is σ-dentable (take either extreme point to be f_ε in 1.13 for each $\varepsilon > 0$.) On the other hand, K is not dentable. Indeed, suppose that $f \in K$. For any integer $n > 0$ choose functions $f_1^n, \ldots, f_n^n$ in K so that $f_i^n(t) = f(t)$ for

$$t \notin [\frac{i-1}{n}, \frac{i}{n}] \text{ and } |f_i^n(t_i^n) - f(t_i^n)| > \frac{1}{2}$$

for some point

$$t_i^n \in (\frac{i-1}{n}, \frac{i}{n}). \text{ Then}$$

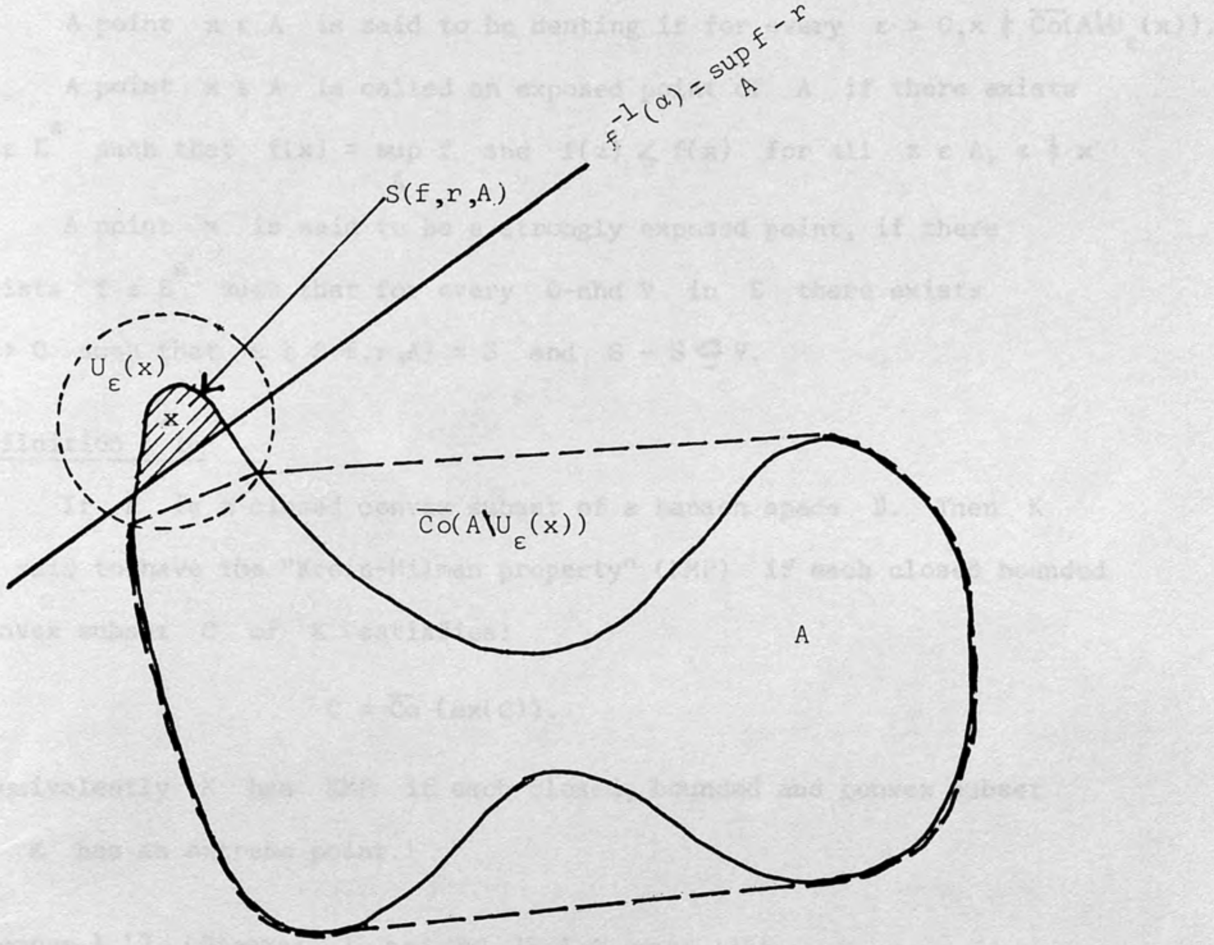
$$\| f_i^n - f \| > \frac{1}{2} \text{ for } i = 1, 2, \ldots, n \text{ and}$$

$$\| \sum_{i=1}^{n} \frac{1}{n} f_i^n - f \| \leqq \frac{2}{n}. \quad \text{It follows that}$$

$f \in \overline{Co}(K \setminus U_{\frac{1}{2}}(f))$ since n was arbitrary. Hence K is not dentable. Consequently $C[0,1]$ is not dentable either.

If A is a bounded subset of ℓ.c.s E a slice of A is a subset of A defined by:

$S(f,r,A) = \{x \in A: f(x) > \sup f - r\}$ where $f \in E^*$, $f \neq 0$ and $r > 0$; see the diagram below



$f^{-1}(\alpha) = \sup_A f - r$
S(f,r,A)
$U_\varepsilon(x)$
x
$\overline{Co}(A\setminus U_\varepsilon(x))$
A

Definition 1.15

A point $x \in A$ is said to be denting if for every $\varepsilon > 0, x \notin \overline{Co}(A \setminus U_\varepsilon(x))$.

A point $x \in A$ is called an exposed point of A if there exists $f \in E^*$ such that $f(x) = \sup_A f$ and $f(z) \lneq f(x)$ for all $z \in A, z \neq x$.

A point x is said to be a strongly exposed point, if there exists $f \in E^*$ such that for every 0-nhd V in E there exists $r > 0$ such that $x \in S(f,r,A) = S$ and $S - S \subseteq V$.

Definition 1.16

If K is a closed convex subset of a Banach space B. Then K is said to have the "Krein-Milman property" (KMP) if each closed bounded convex subset C of K satisfies:

$$C = \overline{Co}\,(ex(C)).$$

(Equivalently K has KMP if each closed, bounded and convex subset of K has an extreme point.)

Theorem 1.17 (Diestel, J. and Uhl JR.J.J. page 138)

Let A be a bounded subset of B.

(i) If $\overline{Co}(A)$ is dentable, then A is dentable

(ii) If A has an exposed point x_o then A is σ-dentable.

(iii) If A has a strongly exposed point x_o then A is dentable.

Proof

(i) Suppose $\overline{Co}(A)$ is dentable and suppose $\varepsilon > 0$. Then there is $x_\varepsilon \in \overline{Co}(A)$ such that $x_\varepsilon \notin \overline{Co}(\overline{Co}(A \setminus U_{\varepsilon/2}(x_\varepsilon)) = Q$. Then $x_\varepsilon \in \overline{Co}\,(A)$ but $x_\varepsilon \notin Q$. Next note that $A \setminus Q$ is not empty; for if $A \subseteq Q$, then closed and

$\overline{Co}(A) \subseteq Q$ since Q is closed and convex. But $x_\varepsilon \in \overline{Co}(A)$ and $x_\varepsilon \notin Q$, a quick contradiction.

Now select $d \in A\backslash Q$. We shall establish that $d \notin \overline{Co}(A\backslash U_\varepsilon(d))$ and thus prove that A is dentable.

To this end, note that $d \in U_{\varepsilon/2}(x_\varepsilon)$. For otherwise

$$d \in A\backslash U_{\varepsilon/2}(x_\varepsilon) \subseteq \overline{Co}(A\backslash U_{\varepsilon/2}(x_\varepsilon)) \subseteq Q;$$

which is impossible since $d \notin Q$. Since $d \in A\backslash Q$ is unspecified otherwise, we have $A\backslash Q \subseteq U_{\varepsilon/2}(x_\varepsilon)$. From this inclusion, the inclusion $A\backslash U_\varepsilon(d) \subseteq Q$ obtains since if $d_o \in A$ and $\|d - d_o\| \geqq \varepsilon$ and $d_o \notin Q$ then $d_o, d \in A\backslash Q$ implies

$$\|d_o - d\| \leq \|d - x_\varepsilon\| + \|x_\varepsilon - d\| < \frac{2\varepsilon}{2} = \varepsilon.$$

Recalling that Q is closed and convex we see that $\overline{Co}(A\backslash U_\varepsilon(d) \subseteq Q$. Since $d \in A\backslash Q$, it follows that $d \notin \overline{Co}(A\backslash U_\varepsilon(d))$.

(ii) If $x_o \in A$ is an exposed point and $\{x_n\} \subseteq A$ is a sequence such that there is a sequence of real $\{\alpha_n\}$ with $0 < \alpha_n \leq 1$ and $\sum_{n=1}^{\infty} \alpha_n = 1$ such that $x_o = \sum_{n=1}^{\infty} \alpha_n x_n$, then one has

$$\sum_{n=1}^{\infty} \alpha_n f(x_o) = f(x_o) = \sum_{n=1}^{\infty} \alpha_n f(x_n) \quad \text{for some } f \in B^*,$$

i.e. $$\sum_{n=1}^{\infty} \alpha_n (f(x_o) - f(x_n)) = 0.$$ Since this last series has non-negative entries, each entry must be zero. Since $\alpha_n > 0$ for all $n \in N$, we see that $f(x_o) = f(x_n)$ for all $n \in N$.

Hence $x_o = x_n$ for all n. It follows that A is σ-dentable.

(iii) Suppose x_o is strongly exposed and suppose $x_o \in \overline{Co}(A\backslash U_\varepsilon(x_o))$ there must be convex sum $\sum_{n=1}^{\infty} \alpha_n x_n$ with $0 < \alpha_n \leq 1$, $\sum_{n=1}^{\infty} \alpha_n = 1$ and $x_n \in A\backslash U_\varepsilon(x_o)$ that are as close to x_o as we please.

Since $f(x_n) < f(x_o)$ for each n, a slight refinement of use in (ii) shows that there must be $\{y_n\}$ in $A \setminus U_\varepsilon(x_o)$ such that $f(y_n) \to f(x_o)$. Hence $\lim_n y_n = x_o$ and this is a contradiction.

Theorem 1.18 (Milman)

Suppose that C is a compact convex set in E and that D is a closed subset of C. If the

$$\overline{\mathrm{Co}}(D) = C \quad \text{then} \quad D \supset ex(C).$$

Theorem 1.19 (Choquet)

Suppose that C is a compact convex metrizable subset of E and suppose that $x \in C$. Then there is a probability measure μ on

(C, Borel subsets of C) such that

$$\mu(ex(C)) = 1 \quad \text{and} \quad \int_C f \, d\mu = f(x) \quad \text{for each} \quad f \in E^*.$$

For the reference of theorem 1.17 and 1.18 see for instance Bourgin [5].

Definition 1.20

A directed set A is a partially ordered set such that for each $\alpha, \beta \in A$ there exists $\gamma \in A$ where,

$$\gamma \geqq \alpha \quad \text{and} \quad \gamma \geqq \beta.$$

Definition 1.21

A net is a function $\alpha \to \lambda(\alpha)$ on a directed set A.

Definition 1.22

A net $\{f_\alpha\}_{\alpha\varepsilon A}$ in a ℓ.c.s E is said to be a Cauchy net if for every $\varepsilon > 0$ there exists $\alpha_o \in A$ such that $\alpha_1, \alpha_2 \geqq \alpha_o$ implies $p(f_{\alpha_1} - f_{\alpha_2}) \leqq \varepsilon$ for every continuous semi-norm p on E.

Definition 1.23

A function $f : X \to E$ is said to be integrable, if and only if there exists a Cauchy net $\{f_\alpha\}_{\alpha \varepsilon A}$ of simple functions such that:

(a) $\lim_\alpha f_\alpha = f \quad \mu.a.e$

(b) $\lim_\alpha \int_X p(f_\alpha - f)\, d\mu = 0$ for every continuous semi-norm p on E.

Let $\{f_\alpha\}_{\alpha \varepsilon A}, \{g_\alpha\}_{\alpha \varepsilon A}$ be Cauchy nets of simple functions such that

$$\lim_\alpha f_\alpha = f \quad \mu.a.e \quad \text{and}$$

$$\lim_\alpha g_\alpha = f \quad \mu.a.e$$

Then $\{f_\alpha - g_\alpha\}_{\alpha \varepsilon A}$ is a Cauchy net. Hence,

$$\lim_\alpha \int_X (f_\alpha - g_\alpha)\, d\mu = 0$$

for $\quad p\int_X (f_\alpha - g_\alpha)\, d\mu \leqq \int p(f_\alpha - g_\alpha)\, d\mu$.

But

$$p(f_\alpha - g_\alpha) = p[(f_\alpha - f) - (g_\alpha - f)] \leqq$$

$$p(f_\alpha - f) - p(g_\alpha - f) \to 0 \text{ as } \alpha \to \infty \quad \text{So}$$

$$\int p(f_\alpha - g_\alpha) \to 0 \text{ as } \alpha \to \infty.$$

In conclusion, the limit: $\lim_{\alpha} \int_X f_\alpha \, d\mu$ exists independently of the choice of $\{f_\alpha\}_{\alpha \in A}$, satisfying (a) and (b).

The definition 1.22 of an integrable function is in agreement with the Bochner definition of integrability when $E = B$. Then the integral of f, $\int f \, d\mu$ is given by $\lim_{\alpha} \int f_\alpha \, d\mu$.

Chapter 2

This chapter is divided into three sections. Section 1 contains historical background. The Uhl [46] results are in section 2. In section 3 we gave a generalization of the results of Uhl [46] to locally convex spaces in which the Radon-Nikodým derivative (RND) exists.

1. One of the most beautiful and best-loved theorem of the theory of vector measures is the Liapounoff convexity theorem which states that the range of a non-atomic vector measure with value in a finite dimensional space is compact and convex. Later in 1945 Liapounoff showed by example that neither the convexity nor compactness holds in general in the infinite dimensional case.

The next step was taken by Halmos [24] who in 1948 gave a simplified proof of Liapounoff's result for the finite dimensional case.

Blackwell [2] in 1951 considered the case of a measure represented by a finite dimensional vector integral and obtained a result similar to that of Liapounoff. A very short proof of Liapounoff's earlier result was given by Lindenstrass [31].

Olech [35] in 1968 considered the case of an unbounded measure and its range is in a finite dimensional vector space and proved that, in the case of non-atomic unbounded vector measure, the range is convex, the closure of the range of such measure does not contain a line and each compact extreme face of the closure of the range is contained in the range.

2. Uhl [46] in 1969 gave a generalization of Liapounoff's result in the case of a vector measure of bounded variation whose value is contained in a Banach space which is either a reflexive space or a separable dual space, and he proved the following theorem.

Theorem 2.1 Uhl [46]

Let B be a Banach space which is either a reflexive space or a separable dual space, If $m : \sum \to B$ is a vector measure of bounded variation, then the range $m(\sum)$ of m is a conditionally compact subset of B. Moreover, if m is non-atomic, then the closure of $m(\sum)$ is compact and convex.

Proof

We follow Uhl's proof.

Let m and B be as in the hypothesis, and for $A \varepsilon \sum$, define $\mu_A = \mu = |m| |_A$ (i.e. μ_A is the variation of the set function m restricted to A, $\mu_A(C) = \mu(C) = |m|(A \cap C)$ on $\sum$).

μ is a countably additive non-negative finite measure on $\sum$, see Dinculeanu [18] page 41.

Clearly $m \ll \mu$, hence in the case that B is a reflexive space or in the case that B is a separable dual space then theorem (4.1.3) and corollary (4.1.5) of Bourgin [5] respectively, guaranteed the existence of a μ-measurable B - valued function f such that $m(S) = \int_S f\, d\mu$ for every $S \varepsilon \sum$.

Select a sequence of simple functions f_n in $L^1_B(\mu)$ converging to f in $L^1_B(\mu)$ norm, this can be done by virtue of corollary (8) page 125 of Dunford and Schwartz [19].

Define T and T_n, $n = 1, 2, 3\ldots$, for $g \in L_K^{\infty}(\mu)$ where $K = R$ or C, by:

$$T(g) = \int_X g\,f\,d\mu \quad \text{and} \quad T_n(g) = \int_X g\,f_n\,d\mu,$$

respectively. T and T_n are linear from the linearity of integral.

$$\left\| \int_X g\,f\,d\mu \right\| \leq \int_X |g|\ \|f\|_1\,d\mu \leq \|g\|_{\infty} \cdot \|f\|_1, \quad \text{by the}$$

Hölder inequality. This implies that T is bounded and the same holds for T_n.

$$\lim_{n \to \infty} \|T - T_n\| = \lim \sup_{\|g\|_{\infty}=1} \left\{ \left\| \int_X (g\,f_n - g\,f)\,d\mu \right\| \right\}$$

$$\leq \lim_{n \to \infty} \int_X |g|\ \|f_n - f\|\,d\mu = \lim_{n \to \infty} \int_X \|f_n - f\|\,d\mu = 0.$$

The range of each T_n is finite dimensional, since each f_n is a simple function. Therefore, each T_n is a compact operator. Moreover, since $\{\chi_S : S \in \Sigma\}$ is contained in the unit ball of $L_K^{\infty}(\mu)$ it is therefore, bounded.

It follows from the compactness of T, that

$$\{m(S) : S \in \Sigma\} = \left\{\int_S f\,d\mu : S \in \Sigma\right\} = \{T(\chi_S) : S \in \Sigma\}$$

is a norm conditionally compact subset of B.

To prove the second statement, assume that m is non-atomic, then clearly μ is non-atomic.

Let $\pi = \{I_n\}$ be a partition of X, i.e. a finite collection of disjoint sets in Σ whose union is X.

The simple function f_π is defined by:

$$f_\pi = \sum_\pi \frac{\int_{I_n} f\, d\mu}{\mu(I_n)} \chi_{I_n} \quad 0/0 = 0, \text{ and}$$

$$m_\pi(S) = \sum_\pi \frac{\int_{I_n} f\, d\mu}{\mu(I_n)} \mu(I_n \cap S).$$

Therefore, by lemma (111.2.15) Dunford and Schwartz [19] page 109,

$$|m|(S) = \int_S \| f \|\, d\mu \quad \text{for each } S \in \Sigma,$$

$$|m_\pi|(S) = \int \| f_\pi \|\, d\mu \quad \text{for each } S \in \Sigma .$$

We have

$$(m - m_\pi)(S) = \int_S (f - f_\pi)\, d\mu . \quad \text{So}$$

$m - m_\pi$ has RND $f - f_\pi$ with respect to μ. Therefore,

$$|m - m_\pi|(S) = \int \| f - f_\pi \|\, d\mu .$$

Define U_π such that $U_\pi(f) = f_\pi$ for each $f \in L^1_B(\mu)$. Then

$$\| U_\pi f \| = \Big\| \sum_{i=1}^{n} \frac{\int_{I_i} f\, d\mu}{\mu(I_i)} \chi_{I_i} \Big\|$$

$$\leq \sum_{i=1}^{n} \int_{I_i} \| f \|\, d\mu . \quad \text{Therefore}$$

$$\| U_\pi \|_1 \leq \| f \|_1 . \quad \text{But}$$

$$\| U_\pi f \|_1 \leq \| U_\pi \| \cdot \| f \|_1 , \text{ So } \| U_\pi \| \leq 1 .$$

Now put $M = \{f\}$, a bounded, compact subset of $L^1_B(\mu)$.

By virtue of theorem (IV.8.18) Dunford and Schwartz [19]

$$\lim_\pi U_\pi f = f \text{ for each } f \varepsilon L^1_B(\mu) \text{ in } L^1_B(\mu) \text{ norm.}$$

Therefore

$$\lim_\pi \int_X \| U_\pi f - f \| \, d\mu = \lim_\pi \int_X \| f_\pi - f \| \, d\mu = 0$$

so the

$$\lim_\pi |(m - m_\pi)|(S) = \lim_\pi \int_S \| f - f_\pi \| \, d\mu = 0 ,$$

where the limit is taken in the Moore-Smith sense after the collection of all partitions is directed by the partial ordering of refinement.

Note that each m_π has values in a finite dimensional subspace of B. Also, since μ is non-atomic it follows that m_π is non-atomic. Hence by the Liapounoff theorem

$$\{m_\pi(S) : S \varepsilon \textstyle\sum\} \text{ is convex.}$$

Let $x, y \varepsilon \overline{m(\sum)}$, α, β be non-negative numbers with $\alpha + \beta = 1$, and $\varepsilon > 0$ be given.

Select $N_1, N_2 \varepsilon \sum$ such that,

$$\| x - m(N_1) \| \leq \varepsilon/2 \text{ and } \| y - m(N_2) \| \leq \varepsilon/2 .$$

Then choose a partition π_o subject to the conditions that

$$\pi_o \geqq \{N_1 - N_2, N_2 - N_1, N_1 \cap N_2, X - (N \cup N_2)\},$$

and

$$\| m - m_{\pi_o} \| < \varepsilon/2 \ , \ \text{so},$$

$$m_{\pi_o}(I_j) = \sum_i \frac{\int_{I_i} f \, d\mu}{\mu(I_i)} \mu(I_i \cap I_j) = \int_{I_j} f \, d\mu = m(I_j), j = 1,2,3 \ldots .$$

Moreover, since the range of m_{π_o} is convex, there exists a set $N_o \in \sum$, such that

$$m_{\pi_o}(N_o) = \alpha \, m_{\pi_o}(N_1) + \beta \, m_{\pi_o}(N_2) = \alpha \, m(N_1) + \beta \, m(N_2).$$

By combining these relationships, we get:

$$\| \alpha x + \beta y - m(N_o) \| = \| \alpha x + \beta y - (\alpha \, m(N_1) + \beta(m(N_2)) +$$

$$m_{\pi_o}(N_o) - m(N_o) \| \leqq \alpha \, \| x - m(N_1) \| + \beta \| y - m(N_2) \|$$

$$+ \| m_{\pi_o}(N_o) - m(N_o) \| \leqq \alpha \, \varepsilon/2 + \beta \, \varepsilon/2 + \varepsilon/2 = \varepsilon \ ,$$

since $\alpha + \beta = 1$ and

$$\| m_{\pi_o}(N_o) - m(N_o) \| \leqq | m_{\pi_o} - m | < \varepsilon/2$$

thus the closure of the range of m is convex.

Corollary 2.2

Under the same hypothesis of theorem 2.1 if the range of m is norm closed, then it is norm compact. If m is non-atomic and its range is closed, then the range of m is compact and convex.

The proof of corollary 2.2 is a direct consequence of theorem 2.1.

Two examples were given by Uhl [46], the first shows that if B is not a reflexive space and not a separable dual space then the closure of the range of a non-atomic measure can fail to be compact and convex. The second, which is due originally to Liapounoff, shows that, the measure may satisfy the hypotheses of theorem 2.1 and fail to have a compact and convex range. We give these two examples for completeness.

Example 2.3

Let,

$X = [0,1]$

$\sum$ = The Borel σ-algebra of subsets of X

λ = The Lebesgue measure on $\sum$,

$L^1_R(\lambda)$ = The Banach space of the class of Lebesgue measurable functions on X.

It is well-known that $L^1_R(\lambda)$ is not a reflexive space and not a separable dual space(see Riesz and Nagy [40]).

Define

$$m : \sum \to L^1_R(\lambda) \text{ by}$$

$$m(A) = \chi_A \text{ where } \chi_A \text{ is the}$$

characteristic function of $A \in \sum$.

(a) m is a non-atomic; this is clear from the definition of m.

(b) It is clear that m is a countably additive vector measure. Since $\|m(A)\| = \|\chi_A\| = \int |\chi_A| d\lambda = \lambda(A)$, m is of a bounded variation.

(c) The closure of the range of m is neither compact nor convex. To prove now the range of m is not conditionally compact, consider the Borel sets

$$A_r = \bigcup_{t=1}^{2^{r-1}} A_{rt}, \quad r = 1,2,\ldots, \text{ where}$$

A_{rt} is the closed interval

$$\left[2(t-1) / 2^r, \ (2t-1) / 2^r \right] \text{ for } t = 1, 2, \ldots, 2^{r-1}.$$

$A_{r1}, A_{r2}, \ldots, A_{r2^{r-1}}$ are set equally apart and have a common measure $1/2^r$. Similarly, for $A_{r+k}, A_{r+k\,1}, A_{r+k\,2} \ldots, A_{r+k\,2^{r+k-1}}$ are equally apart and have a common measure $1/2^{r+k}$. Therefore, only half of the partitions of A_{r+k} intersect with those of A_r. Hence

$\lambda(A_r \cap A_{r+k}) = 2^{r+k-2} \dfrac{1}{2^{r+k}} = \frac{1}{4}$. So,

$\lambda(A_i \cap A_j) = \frac{1}{4}$ if $i \neq j$.

$A_r = (A_r \cap A_{r+k}) \cup (A_r - A_{r+k})$, take λ for both sides, we get:

$\lambda(A_r) = \lambda(A_r \cap A_{r+k}) + \lambda(A_r - A_{r+k})$

$$\frac{1}{2} = \frac{1}{4} + \lambda(A_r - A_{r+k}) \text{ so ,}$$

$\lambda(A_r - A_{r+k}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ i.e. $\lambda(A_i - A_j) = \frac{1}{4}$ and so;

$$\| \chi_{Ai} - \chi_{Aj} \| = \frac{1}{4} \quad \text{for } i \neq j. \text{ Thus}$$

$\{ \chi_{A_r} \} = \{ m(A_r)\}$ is a sequence in the range of m with no norm convergent subsequence (i.e. the range of m is not conditionally compact).

(Note that we can define $A_r = \{t \in X : \sin 2^r \pi t > 0\}$ for each positive integer r. A brief computation shows that $\| \chi_{A_i} - \chi_{A_j} \| = \frac{1}{4}$ for $i \neq j$.)

To show the closure of the range of m is not convex, note that the function; $\frac{1}{2}\chi_X = \frac{1}{2}\chi_{C_1} + \frac{1}{2}\chi_{C_2}$ where $C_1 = [0,\frac{1}{2}]$, $C_2 = [\frac{1}{2},1]$ is a convex combination of members of the range of m. But if $A \in \sum$ is an arbitrary;

$$\| m(A) - \frac{1}{2} \chi_X \| = \| \chi_A - \frac{1}{2}\chi_X\| = \frac{1}{2}\lambda(X-A) + \frac{1}{2}\lambda(A) = \frac{1}{2}.$$

Thus the closure of the range of m is not convex.

(d) $m << \lambda$, but has no RND with respect to λ. To see this, let $m(A) = \int_A f\,d\lambda$ for some $f \in L^1_R(\lambda)$, for all $A \in \sum$. Then the proof of theorem 2.1 would show that the closure of the range of m was compact. But this contradicts (c).

(e) In view of theorem 2.1 this example gives another proof of the fact that the separable space $L^1_R(\lambda)$ is not a dual space.

Example 2.4

This example, constructed by Liapounoff will be given with some modification to show that even if a vector measure m satisfies the hypotheses of theorem 2.1 its range need not be compact or convex.

Let,

$$X = [0, 2\pi]$$

$\sum$ = the Borel σ-algebra of the subsets of X.

λ = the Lebesgue measure on $\sum$.

Let $\{\psi\}_{n=1}^{\infty}$ be a complete orthogonal set in $L_C^2(\lambda)$ where C is the complex plane, such that each ψ_i assumes only the values ∓ 1 and such that $\psi_o = +1$ while

$$\int_0^{2\pi} \psi_n \, d\lambda = 0 \quad \text{for} \quad n > 0.$$

Define I_n on $\sum$ by :

$I_n(A) = 2^{-n} \int_A ((1 + \psi_n)/2)\, d\lambda$, for all $A \in \sum$. Define $m: \sum \to \ell^2$ by $m(A) = (I_o(A), I_1(A), \dots, I_n(A) \dots)$. Then $\| m(A) \|_{\ell^2}$ =

$$\| m(A) \|_{\ell^2} = \sqrt{\sum_{n=0}^{\infty} | I_n(A) |^2}$$

$$= \sqrt{\sum (2^{-n} | \int_A (1 + \psi_i) / 2\, d\lambda |)^2}$$

$$\leq \sqrt{\sum_{n=0}^{\infty} (\frac{1}{2^n} \int_A | (1 + \psi_i)/2 | d\lambda)^2}$$

$$\leq \sqrt{\sum_{n=0}^{\infty} (\frac{1}{2^n} \lambda(A))^2}$$

$$= \lambda(A) \sqrt{\sum_{n=0}^{\infty} (1/2^n)^2}$$

$= 2\lambda(A)$ for each $A \in \sum$. Therefore, m is of bounded variation.

Clearly m is a non-atomic. Since m has its value in the reflexive space ℓ^2, theorem 2.1 guarantees that the closure of the range of m is compact and convex.

Now consider,

$m(X) = (2\pi, \pi/_2, \ldots \pi/_{2^n}, \ldots)$ and suppose there exists an $A \in \sum$ such that $m(A) = m(X)/2$. Then

$$\pi = I_o(A) = \int_A d\lambda = \lambda(A), \text{ and for } n \geq 1$$

$$\pi/2^{n+1} = I_n(A) = 2^{-n}\int_A (1+\psi_n)/2 \; d\lambda$$

$$= \lambda(A \cap U_n)/2^n, \text{ where}$$

$$U_n = \{S \in [0, 2\pi] : \psi_n(S) = +1\}.$$

It follows immediately from this and the fact that $\lambda(U_n) = \lambda(A) = \pi$ that $\lambda(A \cap U_n) = \lambda(A - U_n) = \lambda(U_n - A) = \lambda(X - (U_n \cup A) = \pi/2$ for all $n > 0$.

Define f on X by :

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \notin A \end{cases}$$

Then

$$\int_0^{2\pi} \psi_o f \, d\lambda = \pi - \pi = 0 \text{ and for all } n > 0$$

$$\int_0^{2\pi} \psi_n f\, d\lambda = \lambda(U_n \cap A) + \lambda(X - (U_n \cup A)) - \lambda(A - U_n) - \lambda(U_n - A) = 0.$$

Since $f \not\equiv 0$ this contradicts the fact that ψ_n was complete in $L^2_C(\lambda)$, and shows two things:

Firstly, that even under the hypotheses of theorem 2.1 such a measure need not have convex range.

Secondly, that in view of corallary 2.2 the range of such a measure need not be closed.

Therefore, theorem 2.1 cannot be improved under the current hypotheses and the conclusions of theorem 2.1 fail to be true in general for a measure with values in an arbitrary ℓ.c.s. even with a Banach space.

3. This section contains a generalization of the Uhl theorem 2.1 in a locally convex space with the RND. We shall give this in theorem 2.5 stated below.

<u>Theorem 2.5</u>

If E is a locally convex space and $m: \sum \to E$ is a vector measure of bounded variation and suppose that m has the RND with respect to $\mu_A = \mu = |m|_p\big|_A$ for all $A \in \sum$; then the range of m is conditionally compact in the topology of E. Moreover, if m is non-atomic then the closure of the range of m is convex.

<u>Proof</u>

Let $m : \sum \to E$ be a vector measure of bounded variation. For each $A \in \sum$, and for every semi-norm $p \in \{p_\alpha\}_{\alpha \in \Lambda}$, let

$$\mu = |m|_p\big|_A \quad .$$

(i.e. $\mu(S) = \sup\{\sum_{i=1}^{n} p(m(S_i)) : S_i \subseteq A \in \Sigma, \text{disjoint } S_i \subseteq S \subseteq A \quad 1 \leq i \leq n\}$).

Then μ is a countably additive non-negative finite measure on Σ. Moreoever, $m << \mu$. Since m has the RND with respect to μ there exists $f \in L^1_E(\mu)$ such that:

$$m(S) = \int_S f \, d\mu, \quad \text{for every } S \in \Sigma.$$

The proof from this point onward is similar to the proof of theorem 2.1, allowing for the following changes:

Theorem 2.1	Theorem 2.5
$\|\cdot\|$	p
$\|T_n - T\|$	$\tilde{p}(T_n - T)$
$\|f\|$	$p(f)$
$\int_X \|f\| \, d\mu$	$\int_X p(f) \, d\mu = q(f)$

where $\tilde{p}(T_n - T) = \sup_{\|g\|_\infty = 1} p(\int_X (g f_n - g f) \, d\mu$, bearing in mind that the space in this theorem is ℓ.c.s. rather than a Banach space.

Remark 2.6

According to the example 2.4 the range of a non-atomic vector measure with value in a ℓ.c.s. need not be convex even if it satisfies the hypotheses of theorem 2.5. Thus theorem 2.5 cannot be improved in this way under the current hypotheses.

Remark 2.7

Under the same hypotheses as in Theorem 2.5, if the range of m is closed so that $m(\sum) = \overline{m(\sum)}$, then $m(\sum)$ is compact. If m is a non-atomic and $m(\sum) = \overline{m(\sum)}$, then $m(\sum)$ is convex.

Clearly the Uhl theorem 2.1 is a special case of theorem 2.5, since every reflexive or separable dual Banach space has the RND by virtue of the theorem 4.1.3 and the corollary 4.1.5 of Bourgin [5].

Before we prove the next result, let us recall the following definition.

Definition 2.8

Let $(X, \sum, \mu)$ be a probability space and $m: \sum \to E$ be a vector measure. A measure m is said to have a locally relatively compact (relatively weakly compact) average range if and only if for each $\varepsilon > 0$ there exists $T_\varepsilon \subseteq X$ such that $\mu(X - T_\varepsilon) \leq \varepsilon$ and the set

$$AR(m) = \{ \frac{m(A)}{\mu(A)} : A \in \sum, A \subseteq T_\varepsilon, \mu(A) > 0\}$$

is relatively compact (relatively weakly compact).

A measure m is said to have a locally dentable (σ-dentable) average range if and only if for each $\varepsilon > 0$ there exists $T_\varepsilon \subseteq X$ such that $\mu(X - T_\varepsilon) \leq \varepsilon$ and the set $AR(m)$ is dentable (σ-dentable).

Let E be a quasi-complete ℓ.c.s. with BM property, and let $(X, \sum, \mu)$ be a probability space and $m : \sum \to E$ be a vector measure with bounded average range.

Theorem 2.9 (Saab) [43]

The following assertions are equivalent:

(i) the measure m has a locally compact average range

(ii) the measure m has a locally weakly compact average range

(iii) the measure m has a locally dentable average range

(iv) the measure m has a locally σ-dentable average range

(v) there exists $f : X \to E$ μ-integrable such that

$$m(A) = \int_A f \, d\mu \quad \text{for every } A \in \sum .$$

For the proof of this theorem we refer to Saab [43] theorem 3.2.

Let E be a quasi-complete ℓ.c.s. with BM property, and let $m : \sum \to E$ be a vector measure of bounded average range which satisfies any one of the properties listed in theorem 2.9 then according to theorem 2.5 $\overline{m(\sum)}$ is compact and convex.

We end this chapter by the following corollary:

Corollary 2.10 (Saab)

Let E be a quasi-complete ℓ.c.s. with BM property, let E have the RNP and let $m : \sum \to E$ be a non-atomic vector measure of bounded variation, then $\overline{m(\sum)}$ is compact and convex.

Proof

Let $m : \sum \to E$ be a vector measure of bounded variation and suppose that m has the RNP with respect to $\mu_A = \mu = |m|_p|_A$, for each $A \in \sum$, and for each semi-norm p.

Then μ_A is a countably additive non-negative finite measure on $\sum$. Moreover $m << \mu_A$. Since m has the RNP with respect to μ_A, then by theorem 2.5 $\overline{m(\sum)}$ is compact and convex.

CHAPTER 3

In 1936, in the United States of America, Tamarkin met J.A. Clarkson and suggested that Clarkson looked at differentiability properties of vector valued functions. This was the beginning of the study of RNP and led to Clarkson's fundamental paper [10] in 1936.

The study of the Radon-Nikodým Theorem (RNT) for the Bochner integral and Orlicz Pettis theorem were to re-establish the links between vector measure, and the analytic, geometric and isomorphic theory of the Banach spaces.

The real break-through in the study of RNP as a geometric propery was provided by Maynard [33] in 1973 who in his paper introduced the notion of σ-dentability and characterized Banach space with RNP as space whose bounded sets are σ-dentable. In this paper he showed that the dentability of a set is a property determined by countable subsets. He did not give a complete solution to the question (in which Banach spaces are bounded sets dentable) which was asked by Rieffel (1967). Using Maynard's work as a basis Davis and Phelps 1974 and Huff 1974 completely solved Rieffel's question.

Here we give a brief summary of the results concerning Banach spaces having the RNP, due to different authors.

A Banach space B has the RNP if and only if one of the following is satisfied.

(a) Every closed linear subspace of B has RNP.

(b) Every separable closed linear subspace of B has RNP.

(c) Every bounded subset of B is dentable.

(d) Every closed bounded convex subset of B is dentable.

(e) Every bounded subset of B is σ-dentable

(f) Every non-empty closed bounded subset of B contains an extreme point of its closed convex hull.

(g) Every non-empty closed bounded convex subset of B is the closed convex hull of its denting points (strongly exposed points).

In 1968, Rieffel [39] proved the fundamental RNT for Banach spaces. This result generalized the classical Lebesgue-Nikodym theorem for R^n to arbitrary Banach spaces. Since then, various efforts have been made to extend Rieffel's RNT to a locally convex space (ℓ.c.s) such as, Rieffel [39], Tweddle 1970 [45] Chi [8, 9] Kupka 1972 [30].

Chi 1975 [8] proved in this paper the RNP for the class of ℓ.c.s's having the property BM^* which was defined in Chapter 1.

G. Gilliam examined the RNP for ℓ.c.s with strict Mackey convergence property.

Saab 1978 [43] in his paper used the technique different to that used by Chi [8] to prove the RNT in class of quasi-complete ℓ.c.s with BM property.

Our goal is to give a presentation of the results of Saab [43] using the same technique that he used.

Throughout this chapter (E,τ) will always be a Hausdorff ℓ.c.s with BM property with τ denoting its topology.

Let $(X, \sum, \mu)$ be a probability space and $m : \sum \to E$ be a vector measure.

Definition 3.1

Let K be a closed bounded convex subset of E. The set K is said to have the RNP if for every probability space $(X, \sum, \mu)$, and for every μ-continuous vector measure $m : \sum \to E$ of bounded variation, there exists an integrable function $f : X \to K$ such that:

$$m(A) = \int_A f \, d\mu, \quad \text{for every } A \in \sum.$$

If every bounded convex subset of E has the RNP, then E is said to have the RNP.

Let C be a closed bounded convex subset of (E,τ). Let $M = \overline{\mathrm{Co}}(C \cup -C)$ and let $E_M = \bigcup_{n=1}^{\infty} nM$. It is clear that M is closed convex and bounded. Then we have the following theorem which was originally proved by Saab [43] in the case where E is a quasi-complete ℓ.c.s with BM property.

Theorem 3.2

There exists a norm N on E_M such that the topology induced by (E_M, N) on M coincides with the topology induced by (E,τ) on M.

Proof

We follow Saab's proof.

Since (E,τ) is ℓ.c.s with BM property there exists a sequence $\{V_n\}$ of closed absolutely convex 0-nhds in (E,τ) such that;

1. $V_{n+1} + V_{n+1} \subset V_n$ for every $n \geq 1$

2. $\{V_n \cap (M-M)\}_{n \geq 1}$ form a fundamental system of 0-nhd in $(M-M,\tau)$.

Let τ_1, be the topology on E that $\{V_n\}_{n \geq 1}$ as a fundamental system of 0-nhds.

The topology τ_1 is not in general Hausdorff but the restriction of τ_1 to E_M is Hausdorff, this comes from (2) above.

Let $x \in M$ and let V be a 0-nhd in E, we have to show that $x + V \cap M \supseteq x + V_n \cap M$ for some n. Note that the existence of n such that

$$V_n \cap (M - M) \subseteq V \cap (M - M) \text{ is guaranteed.}$$

Let $y \in (x + V_n) \cap M$ then $y - x \in V_n \cap M - M$ accordingly $y - x \in V \cap (M - M)$, hence $y \in (x + V)$ M. This proves that the restriction of τ to M is coarser than the restriction of τ_1 to M. On the other hand, it is clear that τ restricted to M is finer than τ_1 restricted to M thus τ_1 and τ agree on M

Since M is bounded, for every n there exist $a_n \geq 1$ such that $M \subseteq a_n V_n$.

Let p_n be the gauge functional of V_n. For every $x \in E_m$ define

$$N(x) = \sum_{n=1}^{\infty} \frac{1}{a_n 2^n} p_n(x).$$

Since each p_n is a semi-norm, so is N. For every $x \in E_M$, if $N(x) = 0$, then $p_n(x) = 0$ for every n, this implies that $x = 0$ because τ_1 is Hausdorff on E_M. It follows that N is a norm on E_M.

Let τ_2 be the topology defined by N on E_M. It is clear that $\tau_1|_M$ is coarser than $\tau_2|_M$. Conversely, let $x \in M$, and let

$$B_N(x,\varepsilon) = \{y \in M;\ N(x-y) \leq \varepsilon\}$$

and let $B_k(x,\varepsilon) = \{y \in M\ ;\ p_k(x - y) \leq \varepsilon\}$

It is enough to prove that

$$B_k(x, \frac{1}{2^k}) \subseteq B_N(x, \frac{3}{2^k}) \quad \text{for each } k.$$

Let $y \in B_k(x, \frac{1}{2^k})$. We have $V_1 \supseteq V_2 \supseteq V_3 \ldots$, so $p_1 \leq p_2 \leq p_3 \ldots$

so $p_1(x-y) \leq p_2(x-y) \ldots \leq p_k(x-y) \leq \frac{1}{2^k}$. Also

$$N(x-y) = \sum_{n=1}^{k} \frac{1}{a_n 2^n} p_n(x-y) + \sum_{n=k+1}^{\infty} \frac{1}{a_n 2^n} p_n(x-y).$$

Let $x \in M$, $y \in M$, so $x-y \in M-M$.

Since $p_n(x-y) \leq p_n(x) + p_n(y)$, we have

$p_n(x-y) \leq 2a_n$ as $p_n \leq 1$ on V_n. Therefore,

$$N(x-y) \leq \frac{1}{2^k} \sum_{n=1}^{k} \frac{1}{2^n} + \sum_{n=k+1}^{\infty} \frac{2a_n}{a_n 2^n} = \frac{1}{2^k} \sum_{n=1}^{k} \frac{1}{2^n} +$$

$$\frac{1}{2^k} \sum_{n=0}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^k} [1 + 2] = \frac{3}{2^k}.$$

This proves that τ_1 restricted to M is finer than τ_2 restricted to M and so τ_1 and τ_2 agree on M.

Denote the completion of (E_M, N) by $(\hat{E}_M, N)$.

Corollary 3.3

Let C and M be as in theorem 3.2 then

(i) the set C is dentable (σ-dentable in (E,τ) if and only if C is dentable (σ-dentable) in $(\hat{E}_M, N)$.

(ii) A point $x \varepsilon C$ is a denting point in (E,τ) if and only if x is a denting point in $(\hat{E}_M, N)$.

Proof

Suppose C is dentable in (E,τ). That is for every $\varepsilon > 0$ there exists a point $x_\varepsilon \varepsilon C$ such that $x_\varepsilon \notin \overline{Co}(C \backslash U_\varepsilon(x_\varepsilon))$ which is a subset of M. But the family of x_ε-nhds $U_\varepsilon(x_\varepsilon)$ is the same on M for τ_1 and τ. So C is dentable.

Suppose C is σ-dentable in E. That is for each $\varepsilon > 0$ there exists a point $x_\varepsilon \ \varepsilon \ C$ such that $x_\varepsilon \notin \sigma\text{-}Co(C \backslash U_\varepsilon(x_\varepsilon))$. But the family of x_ε-nhds $U_\varepsilon(x_\varepsilon)$ is the same on M for τ_1 and τ so C is σ-dentable.

Suppose that $x \ \varepsilon \ C$ is a denting point in (E,τ) that is for every 0-nhd V in E, $x \notin \overline{Co}(C \backslash x + V))$. But the family of 0-nhds V is the same on M for τ_1 and τ so x is a denting point of C in $(\hat{E}_M, N)$.

Theorem 3.4

Let $(X, \sum, \mu)$ be a probability space and let C and M be as above. A function $f : X \to C$ is integrable in $(\hat{E}_M, N)$ if and only if f is integrable in (E,τ).

In this case $\int_A f\,d\mu$ on $(\hat{E}_M, N)$ is the same as $\int_A f\,d\mu$ on (E,τ) for every $A \in \sum$.

Note that the integrability is taken in the sense of Definition 1.23.

Proof

Suppose $f : X \to C$ is integrable in $(\hat{E}_M, N)$ then there exists an N-Cauchy net $\{f_\alpha\}_{\alpha \in S}$ of simple functions, $f_\alpha : X \to C$ for each $\alpha \in S$, such that $\lim_\alpha f = f$ μ.a.e and $\lim_\alpha \int_X N(f_\alpha - f)\,d\mu = 0$.

By virtue of theorem 3.2 $\{f_\alpha\}_{\alpha \in S}$ is a p-Cauchy net and $\lim_\alpha f_\alpha = f$ u.a.e in (E,τ).

To complete the proof we have to show that $\lim_\alpha \int_X p(f_\alpha - f)\,d\mu = 0$ for every continuous semi-norm p on (E,τ). Note that although the injection $J : (E_M, N) \to (E,\tau)$ is not necessarily continuous, its restriction to M is continuous by theorem 3.2.

Consider the net $h_\alpha = p(f_\alpha - f)$; this is a net of real valued functions and each $f_\alpha - f$ has value in M which is a bounded set, so h_α is a net of uniformly bounded integrable functions which tend to zero μ.a.e.

By the bounded convergence theorem we have $\lim_\alpha \int_X p(f_\alpha - f)\,d\mu = 0$. So f is integrable in (E,τ).

Conversely, suppose that $f : X \to C$ is integrable in (E,τ). Consider the sequence $\{p_n\}_{n=1}^\infty$ of semi-norms which define the topology τ_1 on E_M. As before $V_1 \supseteq V_2 \supseteq V_3 \dots$ and $p_1 \leq p_2 \leq p_3 \dots$.

Since $f(x) \in C$, $f(x) \in E_M$. We have to show first for each n there exists a p_n - Cauchy net of simple functions $\{\phi_t^k\}_{k\geq 1}$ with values in E_M such that $p_n(\phi_t^k - f) \to 0$ μ.a.e as $k \to \infty$.

Since f is measurable in (E,τ) there exists a net of simple functions $\sum_{i=1}^{t} a_{im}\chi_{E_{im}}(x)$ such that

$$p_n\Big(\sum_{i=1}^{t} a_{im}\chi_{E_{im}}(x) - f(x)\Big) \to 0 \quad \mu.\text{a.e}$$

as $m \to \infty$.

We may suppose that the sets E_{im} are pairwise disjoint and not of μ-zero measure.

For each i,m there exists $b_{im} \varepsilon \hat{E}_M$ such that

(1) $p_n(a_{im} - b_{im}) \leq \inf\limits_{x\varepsilon E_{im}} (f(x) - a_{im}) + \frac{1}{m}$. Since $f(x) \varepsilon E_M$ μ.a.e So

$\{f(x) : x \varepsilon E_{im}\} \cap E_M \neq \phi$, then for any x and fixed m, we have

(2) $p_n\{f(x) - \sum_{i=1}^{t} b_{im}\chi_{E_{im}}(x)\} \leq p_n(f(x) - \sum_{i=1}^{t} a_{im}\chi_{E_{im}}(x))$

$$+ p_n\Big(\sum_{i=1}^{t}(a_{im} - b_{im})\chi_{E_{im}}(x)\Big)$$

from (1) $p_n(a_{im} - b_{im}) \leq p_n(f(x) - a_{im}) + \frac{1}{m}$ for every $x \varepsilon E_{im}$ so,

$$\sum_{i=1}^{t} p_n(a_{im} - b_{im})\chi_{E_{im}}(x) \leq p_n(f(x) - \sum_{i=1}^{t} a_{im}\chi_{E_{im}}(x)) + \frac{1}{m}, \text{ therefore}$$

$$p_n(f(x) - \sum_{i=1}^{t} a_{im}\chi_{E_{im}}(x)) + p_n\Big(\sum_{i=1}^{t} a_{im} - b_{im})\chi_{E_{im}}(x)\Big)$$

$$\leq 2\, p_n(f(x) - \sum_{i=1}^{t} a_{im}\chi_{E_{im}}(x)) + \frac{1}{m} \to 0 \text{ as } m \to \infty.$$

So $p_n(f(x) - \sum_{i=1}^{t} b_{im} \chi_{E_{im}}(x)) \to 0$ as $m \to \infty$ for each n.

Now denote the sequence

$\sum_{i=1}^{t} b_{ik} \chi_{E_{ik}}(x)$ by $\phi_t^k(x)$. This sequence is bounded and has its value in the closed bounded and convex subset C of E. Moreover $\lim_{k\to\infty} p_n(\phi_t^k(x) - f(x)) = 0$ μ-a.e. So by virtue of the bounded convergence theorem, we have

$$\lim_{k\to\infty} \int_X p_n(\phi_t^k(x) - f(x))\, d\mu = \int_X \lim_{k\to\infty} p_n(\phi_t^k(x) - f(x))\, d\mu = 0$$

for every $n \geqq 1$.

Choose:

$$\phi_{i_1}^1 \quad \text{with} \quad \int_X p_1 (\phi_{i_1}^1 - f)\, d\mu < 1 \quad \text{and}$$

$$\phi_{i_2}^2 \quad \text{with} \quad \int_X p_2 (\phi_{i_2}^2 - f)\, d\mu < \frac{1}{2}$$

$$\vdots$$

$$\phi_{i_n}^k \quad \text{with} \quad \int_X p_n (\phi_{i_n}^k - f)\, d\mu < \frac{1}{n}$$

take $f_k = \phi_{i_k}^k$,

f_k is a sequence of simple functions with values in C for every k i.e. $f_k : X \to C$. For $k \geqq n$, we have

$$\lim_{k\to\infty} \int_X p_n(f_k - f) = \lim_{k\to\infty} \int_X p_n(\phi_{i_k}^k - f) \leqq \lim_{k\to\infty} \frac{1}{k} \to 0.$$

If $k < n$ and since $p_k \leq p_n$

so $\lim\limits_{k\to\infty} \int_X p_n(f_k - f) \leq \lim\limits_{k\to\infty} \int p_k(f_k - f) \to 0.$

Since mean convergence implies a.e convergence of a sub-sequence, there is a sub-sequence

(1) $f_1^{(1)}, f_2^{(2)}, \ldots \to f$ a.e p_1 and there a sub-sequence

which converge to f a.e under p_2 call it

(2) $f_1^{(2)}, f_2^{(2)}, \ldots \to f$ a.e p_2,

and there is a sub-sequence of (2) which converge to f a.e under p_3 call it

$f_1^{(3)}, f_2^{(3)} \ldots\ldots \to f$ a.e p_3

and consequently, there is a sub-sequence of $(n-1)$ which converges to f a.e under p_n

Call it

$f_1^{(n)}, f_2^{(n)} \ldots\ldots \to f$ a.e. p_n

From the above we have the following sub-sequence

$f_1^{(1)}\ f_2^{(1)}\ f_3^{(1)} \ldots\ldots \to f$ a.e. p_1

$f_1^{(2)}\ f_2^{(2)}\ f_3^{(2)} \ldots\ldots \to f$ a.e. p_2

.
.
.
.

$f_1^{(n)}\ f_2^{(n)}\ f_3^{(n)} \ldots\ldots \to f$ a.e. p_n

with $\{f_i^{(n-1)}\}_{i=1}^{\infty}$ is a sub-sequence of $\{f_i^{(n)}\}_{i=1}^{\infty}$.

Choose $g_n = f_n^{(n)} : X \to C$.

g_n is a sequence of simple functions which converge μ.a.e to f for the topology τ_1 . This proves that $g_n \to f$ μ.a.e in $(\hat{E}_M,N)$ and hence f is measurable in $(\hat{E}_M,N)$. Since f is bounded in $(\hat{E}_M,N)$ therefore f is integrable in $(\hat{E}_M,N)$ from this directly by Definition 1.23.

Corollary 3.5

Let C and M as above then C has the RNP in (E,τ) if and only if C has the RNP in $(\hat{E}_M,N)$.

Now using the corollary 3.3 , 3.5 together with the results of Huff [25] stated below we can easily prove theorem 3.7, so first we give Huff's result.

Lemma 3.6 (Huff) [25]

For a Banach space B the following statements are equivalent.

(i) B has the RNP

(ii) Every bounded subset of B is dentable

(iii) Every bounded subset of B is σ-dentable.

Theorem 3.7

Let C be a closed bounded subset of E, then the following statements are equivalent.

(i) The set C has the RNP.

(ii) The set C is subset dentable.

(iii) The set C is subset σ-dentable.

Proof

(i) $\Rightarrow$ (ii) :

Let $M = \overline{Co}(C \cup -C)$ and $E_M = \bigcup_{n=1}^{\infty} nM$ and $(\hat{E}_M, N)$ be as defined in theorem 3.2 and suppose C has the RNP in (E,τ). Then by corollary 3.5 C has RNP in $(\hat{E}_M, N)$ and so C is subset dentable by lemma 3.6 therefore C is subset dentable in (E,τ) by the corollary 3.3

(ii) $\Longrightarrow$ (iii) is direct from definitions.

(iii) $\Longrightarrow$ (i) :

Let M , E_M, $(\hat{E}_M, N)$ as in theorem 3.2. Suppose C is subset σ-dentable in (E,τ) then by corollary 3.3 C is subset σ-dentable in $(\hat{E}_M, N)$. Therefore by lemma 3.6 C has the RNP in $(\hat{E}_M, N)$. By applying corollary 3.5 we get the result.

Phelps [37] proved the following: Let B be a Banach space, then every bounded subset of B is dentable if and only if every bounded closed convex subset of B is the closed convex hull of its strongly exposed points.

Now by using Phelps result and corollary 3.5 we can prove the following.

Theorem 3.8

Let C be a closed bounded convex subset of (E,τ), then the following statements are equivalent:

(i) The set C has the RNP.

(ii) Every closed convex subset of C is the closed convex hull of its denting points.

Proof

(i) $\Longrightarrow$ (ii)

Let $M = \overline{Co}(C \cup -C)$ and consider $M \subseteq (\hat{E}_M, N)$.

Let C_1 be a closed convex subset of C. Then C_1 has the RNP in (E,τ) and therefore C_1 has the RNP in $(\hat{E}_M, N)$. So by Phelps statement above C_1 is the closed convex hull of its denting points in $(\hat{E}_M, N)$ so theorem 3.2 and corollary 3.3 finish the proof.

(ii) → (i) is immediate from the definitons and theorem 3.7 .

Now we will use the well known results in Banach space and what we did before to deduce the RNT which also appears in chapter 2.

Theorem 3.9 RNT

Let $(X, \sum, \mu)$ be a probability space and $m : \sum \to E$ be a vector measure with bounded average range $AR(m)$ then the following statements are equivalent.

(i) The measure m has a locally relativelycompact average range.

(ii) The measure m has a locally relatively weakly compact average range.

(iii) The measure m has a locally dentable average range.

(iv) The measure m has a locally σ-dentable average range.

(v) There exists a function $f : X \to E$ which is integrable such that

$$m(A) = \int_A f \, d\mu : \text{ for every } A \in \sum .$$

Proof

We reduce the proof to the case of Banach space by considering

$M = \overline{Co}(AR(m) \cup - AR(m))$, and every thing can be studied inside M considered as a subset of the Banach space $(\hat{E}_M, N)$.

Lemma (Smulian)

A convex subset C of a Banach space B is weakly compact if and only if every decreasing sequence of non-empty closed convex subsets of C has a non empty intersection.

For the proof of this lemma see Dunford and Schwartz [19] page 433.

Proposition 3.10

Let C and M as in theorem 3.2 then C is weakly compact in (E,τ) if and only if C is weakly compact in $(\hat{E}_M, N)$.

Proof

The necessity of the condition is immediate from theorem 3.2. For the sufficiency, let $C_1, C_2, C_3 \ldots$ be a decreasing sequence of non-empty closed convex subsets of C, then by hypotheses and Smulian's lemma this sequence has a non-empty intersection in $(\hat{E}_M, N)$ and since the set of the sequence $\{C_i\}_{i=1}^{\infty}$ all lie in M and M has the same topology induced by τ by virtue of theorem 3.2 so $\{C_i\}_{i=1}^{\infty}$ has non-empty intersection in (E,τ) and so C is weakly compact in (E,τ).

The following result shows that the Dunford-Pettis-Phillips theorem is valid in the class of ℓ.c.s's E.

Theorem 3.11

For every weakly compact operator $W : L^1_{[0,1]}(\lambda) \to (E,\tau)$ there exists $g : [0,1] \to (E,\tau)$ λ-integrable (λ is the lebesgue measure on $[0,1]$.) such that $W(f) = \int_0^1 f\, g\, d\,\lambda$ for every f in $L^1_{[0,1]}(\lambda)$, and in particular W sends weakly relatively compact sets into relatively τ - compact sets.

The proof of this theorem is step by step the same as proposition 3.4 of Saab [44].

Definition and Notations 3.12

Let $\sum$ be the σ-algebra on a set X. If M_μ denotes the μ-completion of $\sum$ for each probability measure μ on $\sum$ then let

$$U(\textstyle\sum) = \bigcap_{\mu}\{M_\mu : \mu \text{ is a probability measure on } \textstyle\sum\}$$

$U(\sum)$ is known as the σ-algebra of universally measurable subsets of X relative to $\sum$.

A σ-algebra $\sum$ is called universal if and only if $U(\sum) = \sum$.

Lemma 3.13

Let C be a closed convex separable subset of a Banach space. Then $ex(C) \in U(B(C))$ where $B(C)$ is denoted by the Borel σ-algebra of subsets of C. For the proof of this lemma see Bourgin [5] theorem 6.2.6.

Edgar [20] established a representation theorem of Choquet type for a bounded convex separable subset C of a Banach space B when C has RNP. We give now the statement of the Edgar theorem and we refer to Edgar for the proof.

Theorem 3.14 (Edgar)

Let B be a Banach space with RNP, let C be a closed bounded, separable convex subset of B. Then for every $a \in C$, there exists a probability measure μ on the universal measurable subsets of C such that $a = \int x \, d\mu$ as x is Bochner integrable, and $\mu(ex(C)) = 1$. In particular C is the closed convex hull of $ex(C)$.

We are going to show that the above theorem is also valid in the ℓ.c.s's, with BM property.

Theorem 3.15

Let C be a closed bounded convex separable subset of (E,τ) having the RNP. Then for every $a \in C$ there exists a probability measure μ on the universally measurable subsets of C, such that

$$\mu(ex(C)) = 1 \quad \text{and} \quad \int_C x \, d\mu = a \quad \text{in } (E,\tau).$$

Proof

Let $M = \overline{Co}(C \cup -C)$ and consider C in $(\hat{E}_M, N)$. Since C has the RNP in (E,τ), then the set C has the RNP in $(\hat{E}_M, N)$. Now by the Edgar theorem for every $a \in C$ there exists a probability measure μ defined on the universally measurable subset of C such that $\mu(ex(C)) = 1$ and $a = \int_C x \, d\mu$ in $(\hat{E}_M, N)$. Therefore by theorem 3.5 $a = \int_C x \, d\mu$ in (E,τ). Note that $ex(C)$ is measurable by virtue of lemma 3.13.

The uniqueness of the measure in Edgars theorem is given by Bourgin and Edgar [6]. Before we give this theorem we have to give some definitions.

A finite dimensional simplex S is usually defined to be the convex hull of finitely many affinely independent points (A set Y is affinely independent provided no point $y \in Y$ is in the linear variety generated by $Y \setminus \{y\}$.). It has the property that each point of S is a convex combination of the vertices of S in a unique way.

We avoid the direct generalization of the finite dimensional simplex by using an algebraic characterization of finite dimensional simplices which requires no modification in order to serve as a general definition.

To specify the general definition of simplex we have to give the following terminology.

Say that a subset S of a vector space V over R is in general position if S lies in a hyperplane of V which misses the origin. (Note that for any subset S of V, $S \times \{1\} \subset V \times R$ is in general position.)

If S is a convex subset of V and S in general position let

$$C(S) = \{ts : t \geqq 0, s \varepsilon S\} ,$$

denote the cone over S with vertex 0. Define a partial ordering on span(S) by $x \leqq y$ iff $y - x \varepsilon C(S)$.

Definition 3.16

Let S be a convex subset of a vector space V over R and assume that S is in general position, (if it is not, replace S by $S \times \{1\}$ and V by $V \times R$.) Then S is a simplex if and only if the partial ordering $\leqq$ on span (S) with positive cone $C(S)$ is a lattice ordering on span (S). That is, S is a simplex if and only if each two elements of span (S) have a least upper bound-equivalently each two elements of span (S) have a greatest lower bound in span (S).

We have to prove that the finite dimensional simplices may in fact be characterized as in definition 3.16.

Proposition 3.17

Suppose that span (S) has finite dimension n. Then S is a simplex if and only if it is the convex hull of n linearly independent points. Equivalently, S has exactly n extreme points.

Proof

We can assume without loss of generality that $V = \mathrm{span}(S)$. Suppose S has exactly n extreme points, $x_1, x_2, \ldots, x_n$; since S is the convex hull of its extreme points and since S generates the n-dimensional space V, these points must be linearly independent, and hence they form a basis for V. Choose a basis $f_1, f_2, \ldots, f_n$ for V^* such that $f_i(x_j) = \delta_{ij}$. The map $T : V \to R^n$ defined by

$$T(x) = (f_1(x), \ldots , f_n(x))$$

is linear, one - to - one and onto, and carries x_i onto the n unit vectors in R^n and $C(T(S))$ is the cone of non-negative elements in R^n. This cone induces a lattice ordering in R^n, so $T(S)$ is a simplex; it follows that S itself is a simplex. To finish the proof, suppose that S is a simplex and note that S is the convex hull of its extreme points. Since S generates V, it must have at least n extreme points; we will show that it has exactly n extreme points. Suppose that the points $y_1, y_2, \ldots, y_{n+1}$ are distinct extreme points of V. Since V is n-dimensional, there exist numbers α_i, not all zero, such that $\sum_{i=1}^{n+1} \alpha_i y_i = 0$. Partition the integers from 1 through $n+1$ into the sets D and N where $i \in D$ if $\alpha_i \geq 0$, $i \in N$ otherwise. Then if $\alpha = \Sigma_D \alpha_i$ we have $\alpha > 0$ and (since $f(S) = 1$ for some $f \in V^*$) $\Sigma \alpha_i = 0$ so $\alpha = - \Sigma_N \alpha_i$. Finally, let $x = \Sigma_D \alpha^{-1} \alpha_i y_i = \Sigma_N (-\alpha)^{-1} \alpha_i y_i$. Since these are convex combinations, we have represented an element x by two different measures on S which have support contained in $ex(S)$. It follows from Choquet's uniqueness theorem that S is not a simplex.

Theorem 3.18 (Edgar [20])

Let C be a closed and bounded separable convex subset of a Banach space B with the RNP. Then C is a simplex if and only if for each $a \in C$ there is a unique probability measure μ such that $\mu(ex(C)) = 1$ and $a = \int_C x \, d\mu$ (a a Bochner integral). From the above theorem we can prove now the following

Theorem 3.19

Let C be a closed bounded convex separable subset of (E,τ) having the RNP. Then the following assertions are equivalent:

(i) the set C is a simplex.

(ii) for every $a \in C$ there exists a unique probability measure μ on the universally measurable subsets of C such that

$$a = \int_C x \, d\mu \quad \text{and} \quad \mu(ex(C)) = 1.$$

Proof

(i) $\to$ (ii)

Let $M = \overline{Co}(C \cup -C)$ and consider C in $(\hat{E}_M, N)$. Since C has the RNP in (E,τ), C has the RNP in $(\hat{E}_M, N)$. Then by theorem 3.18 for every $a \in C$ there exists a unique probability measure μ on the universally measurable subsets of C such that $a = \int_C x \, d\mu$, and $\mu(ex(C)) = 1$ and so by virtue of theorem 3.5 we finish the proof.

(ii) $\to$ (i)

Suppose that for every $a \in C$ there exists a unique probability measurable subsets of C such that $\int x \, d\mu = a$ and $\mu(ex(C)) = 1$ in $(\hat{E}_M, N)$, so by theorem 3.18 C is a simplex in $(\hat{E}_M, N)$ and so C is a simplex in (E,τ).

C H A P T E R 4

This chapter deals with the Bishop-Phelps property (BPP) which is related to the RNP in the class of quasi-complete ℓ.c.s. having the property BM.

In section 1 of this chapter we give the definition of BPP for a Banach space and some well known results for Banach space which are proved by Bourgain [4].

In section 2 we extend the definition of BPP given by Bourgain to the class of quasi-complete ℓ.c.s. with BM property, and we prove that if E has BPP then E has RNP.

In section 3 a new definition of BPP on a class of quasi-complete ℓ.c.s. with BM property is given to help us prove the equivalence of BPP in these spaces. We state the definition of BPP which is given by Egghe [22].

We prove that the extension of the Bourgain definition of BPP gives the Egghe definition and the latter definition gives our definition of BPP. At the end of this section we prove that RNP gives the BPP in the sense of our definition by using the same technique that Saab [43] used.

In section 4 we give the extension of Saab's results on cross product of two quasi-complete ℓ.c.s's with BM property. In the end of this section we give a theorem that every barrelled space has BM property.

1. This section contains the definition of BPP for a Banach space amd some well-known results of the BPP on Banach space which are given by Bourgain [4].

Notation 4.1

For any Banach space $(B, \| . \|)$, let

a) $U(B)$ denote the closed unit ball of B and recall that for $x \varepsilon B$ and $\varepsilon > 0$ $U_\varepsilon(x)$ denotes the open ε-ball centered at x, while $U_\varepsilon[x]$ denotes its closure;

b) $S(B)$ denotes the unit sphere of B i.e.

$$S(B) = \{x \varepsilon B : \| x \| = 1\}.$$

c) For any Banach space X let $L(B,X)$ denote the Banach space of bounded linear operators on B into X, with usual norm

$$\| T \| = \sup\{\| T(x) \| : x \varepsilon U(B)\}.$$

Definition 4.2

Let D be a closed and bounded subset of Banach space B. D has the BPP if whenever X is a Banach space and $T \varepsilon L(B,X)$ then there exists a sequence $\{T_n : n = 1,2,...\}$ in $L(B,X)$ and a sequence $\{x_n : n = 1,2 ...\}$ in D such that for each n, both

i) $\| T_n \| < \frac{1}{n}$; and

ii) $\|(T_n + T)(x_n)\| = \sup_y : \{ \| (T_n + T)(y)\| : y \varepsilon D\}$.

Let B be a Banach space and $K \subseteq B$ be a closed, convex subset of B.

Lemma 4.3

The following statement on K are equivalent

(i) K has the RNP.

(ii) Each closed bounded convex separable subset of K has RNP.

(iii) K is subset σ-dentable.

(iv) K is subset dentable.

(v) Each closed bounded convex subset of K is dentable.

For the proof of this lemma see Bourgin [5] theorem 2.3.6 page 31.

Theorem 4.4 (Asplund, Namioka, Bourgain)

Suppose that $\varepsilon > 0$. Let J, K_o, and K_1 be closed bounded convex subsets of B such that

(i) $J \subset \overline{co}(K_o \cup K_1)$

(ii) $K_o \subseteq J$ and diameter $(K_o) < \varepsilon$

(iii) $J \setminus K_1 \neq \phi$

Then there is a slice of J which contains a point of K_o and is of diameter less than ε.

Proof See Bourgin [5] page 51 theorem 3.4.1.

Lemma 4.5

Let K be a closed bounded convex non-dentable subset of B. Then there is an $\varepsilon > 0$ such that whenever D_o and D_1 are subsets of K for which

(i) diameter $(D_o) \leqq \varepsilon$; and

(ii) $K = \overline{Co}(D_o \cup D_1)$,

then $K = \overline{Co}(D_1)$.

Proof

Pick $\varepsilon > 0$ so that each slice of K has diameter at least 2ε. Let $K_o = \overline{Co}(D_o)$ and $K_1 = \overline{Co}(D_1)$. Of course, diameter $(K_o) \leqq \varepsilon$. If also $K \setminus K_1 \neq \phi$ then there is a slice of K which contains a point of K_o of diameter less than 2ε by theorem 4.4. From this impossibility we conclude that $K = \overline{Co}(D_1)$.

Proposition 4.6

Let C be a separable, closed, bounded and convex subset of B. If C has BPP then C is dentable.

Proof

We follow the proof of J. Bourgain [4].

Assume that C is not dentable. Without loss of generality, assume $C \subseteq U(B)$. According to Lemma 4.5 there is an $\varepsilon > 0$ such that whenever D_o and D_1 are subsets of C with $C = \overline{Co}(D_o \cup D_1)$ and the diameter of $(D_o) \leqq 2\varepsilon$ then $C = \overline{Co}(D_1)$.

We can show by induction that

(1) $C = \overline{Co}\,(C \setminus \bigcup_{i=1}^{k} U_\varepsilon[y_i])$ for any finite subset $\{y_i\}_{i=1}^{k}$ of C.

Let $Z = \text{span } C$ and denote by $\{z_n : n=1,2,\dots\}$ a dense sequence in Z. For each n let $\langle z_n \rangle$ be a shorthand notation for the one dimensional space and define a nonlinear operator

$$\psi : B \to \ell_2 \text{ by}$$

$$\psi(x) = (\|x\|, \frac{1}{2} d(x, \langle z_1 \rangle), \ldots, \frac{1}{2^n} d(x, \langle z_n \rangle), \ldots) \varepsilon\, \ell_2$$

where $d(x, \langle z_n \rangle)$ denote the norm distance from B to $\langle z_n \rangle$.

Let $|||x||| = \|\psi(x)\|_2$. ($\|\cdot\|_2$ is the usual ℓ_2 norm.)

Evidently $|||\cdot|||$ defines an equivalent norm on B. Since C has the BPP, the identity operator

$$I : (B, \|\cdot\|) \to (B, (|||\cdot|||))$$

may be perturbed by an operator $T \varepsilon L((B, \|\cdot\|), (B, |||\cdot|||))$ with $\|T\| < \frac{\varepsilon}{4}$ in such a way that for some $x_o \varepsilon C$

(2) $\quad |||(I + T)(x_o)||| = \sup\{|||(I + T)(y)||| : y \varepsilon C\}.$

Pick j so that $\|x_o - z_j\| < \frac{\varepsilon}{4}$. Let $\{y_1, y_2, \ldots, y_k\}$ be a finite $\varepsilon/8$ net for $\{y \varepsilon \langle z_j \rangle : \|y\| \leq 1+\varepsilon\}$, (i.e. for each $1 \leq i \leq k$ there is $1 \leq j \leq k$ such that $\|y_i - y_j\| < \varepsilon/8$) and observe that,

(3) $\quad D = \{y_i \varepsilon C : d(y, \langle z_i \rangle) < \frac{7\varepsilon}{8}\} \subseteq \bigcup_{i=1}^{k} U_\varepsilon[y_i].$

Thus from (1) $C = \overline{co}(C \setminus D)$. It follows that

$$2|||(I+T)(x_o)||| = \sup\{|||(I+T)(x_o) + (I+T)(y)|||: y \varepsilon C \setminus D\}.$$

If $\{v_i : i = 1,2,\ldots\}$ is a sequence in $C \setminus D$ such that

$$2|||(I+T)(x_o)||| = \lim_n |||(I+T)(x_o) + (I+T)(v_n)|||$$

then as we show next,

$$(4)\qquad d((I+T)(x_o), \langle z_j \rangle) = \lim_n d((I+T)(v_n), \langle z_j \rangle).$$

Indeed, for any point p of B and a sequence of points

$\{p_n : n=1,2,\ldots\}$ in B with $\|\psi(p_n)\|_2 \leq \|\psi(p)\|_2$ for any n,

and $\lim_n \|\psi(\frac{1}{2}p + \frac{1}{2}p_n)\|_2 = \|\psi(p)\|_2$ we have

$$\|\psi(p)\|_2 = \lim_n (\psi(\tfrac{1}{2}p + \tfrac{1}{2}p_n)\|_2 \leq \lim_n \|\tfrac{1}{2}\psi(p) + \tfrac{1}{2}\psi(p_n)\|_2 \leq \|\psi(p)\|_2$$

and from the uniform convexity of the ℓ_2 norm, it follows that

$\|\psi(p_n) - \psi(p)\|_2$ converges to 0 as $n \to \infty$ in particular, the $(j+1)^{st}$ coordinate of $\psi(p_n)$ converges to the $(j+1)^{\underline{st}}$ coordinate of $\psi(p)$.

Now let p be the point $(I+T)(x_o)$ and p_n the point $(I+T)(v_n)$ for each n, then formula (4) follows.

On the other hand

$$(5)\quad d((I+T)(x_o), \langle z_j \rangle) \leq \|(I+T)(x_o) - x_o\| + d(x_o, \langle z_j \rangle) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

while for each n

$$(6)\quad d((I+T)(v_n), \langle z_j \rangle) \geq d(v_n, \langle z_j \rangle) - \|(I+T)(v_n) - v_n\| \geq \frac{7\varepsilon}{8} - \frac{\varepsilon}{4} = \frac{5\varepsilon}{8}.$$

The combination of (5) and (6) is impossible in the light of (4). This contradiction proves the proposition.

The next theorem shows that if K has BPP then K has RNP. The proof of this theorem is a direct consequence of lemma 4.3 and proposition 4.6.

Theorem 4.7

Let K be a closed, bounded convex subset of a Banach space B. If each separable closed bounded convex subset of K has BPP then K has RNP.

Proof

Let K be a closed, bounded, convex subset of B, and let D be any separable closed bounded, convex subset of K and suppose that D has BPP Then by proposition 4.6 D is dentable and this is true for any separable closed bounded convex subset of K so K has RNP by lemma 4.3(v).

We will pass to the proof of the converse.

To prove the converse of theorem 4.7 we need the following two lemmas used by J. Bourgain [4] with essentially the same proof.

Lemma 4.8

Let $\{V_n\}_{n=0}^{\infty}$ be a sequence of nonempty sets in B satisfying the following condition:

There is an $\varepsilon > 0$ and a $k > 0$ such that for each $z \varepsilon \mathrm{Co}(V_n)$ and each $x \varepsilon B$, $d(z, \mathrm{Co}(V_{n+1} \setminus U_\varepsilon(x))) < k2^{-n}$. Then the set

$$A = \bigcap_{n=0}^{\infty} \overline{\bigcup_{j \geq n} \mathrm{Co}(V_j)}$$

is nonempty and not dentable.

Proof

First we prove that $Co(V_n) \subseteq A + U_{k2^{-n+1}}(0)$. If $z \in Co(V_n)$, then there exists a sequence $\{z_j\}_{j \geq n}$ such that $z_n = z$, $z_j \in Co(V_j)$ and $\| z_j - z_{j+1} \| < k2^{-j}$.

Clearly $\{z_j\}_{j \geq n}$ converge to some point $a \in A$ and furthermore $\| z - a \| < k2^{-n+1}$. So this proves that

$$Co(V_n) \subseteq A + U_{k2^{-n+1}}(0).$$

Now we show that if $x \in A$, then $x \in \overline{Co}(A \setminus U_{\varepsilon/2}(x))$. Let $x \in A$ and let $0 < \delta < \varepsilon$.

Take $n \in N$ such that $k2^{-n+2} < \delta$.

There is some $j \geq n$ and some $z \in Co(V_j)$ satisfying $\| x - z \| < \delta/2$

Because $d(z, Co(V_{j+1} \setminus U_\varepsilon(x))) < k2^{-j}$ and

$$V_{j+1} \setminus U_\varepsilon(x) \subseteq (A + U_{k2^{-j}}(0)) \setminus U_\varepsilon(x) \subseteq (A \setminus U_{\varepsilon/2}(x)) + U_{k2^{-j}}(0)$$

it follows that $d(z, Co(A \setminus U_{\varepsilon/2}(x))) < k2^{-j+1} < \delta/2$ and hence

$d(x, Co(A \setminus U_{\varepsilon/2}(x))) < \delta$. Since $\delta > 0$ can be taken arbitrarily small, $x \in \overline{Co}(A \setminus U_{\varepsilon/2}(x))$. Thus A is not dentable.

Lemma 4.9

Let D be a nonempty, closed and absolutely convex subset of B, and D contained in the unit ball. Assume that every nonempty subset of D is dentable. Let X be a Banach space. Let $\varepsilon > 0$

be given and define

$$A_\varepsilon = \{T \varepsilon L(B,X); S(T,\eta) \subseteq U_\varepsilon(x) \cup U_\varepsilon(-x)$$

for some $\eta > 0$ and $x \varepsilon B\}$, where

$$S(T,\eta) = \{x \varepsilon D ; \|T(x)\| \geqq \sup_{y \varepsilon D}\{\|T(y)\|\} - \eta\}$$

Then A_ε is dense in $L(B,X)$. Moreover, if $\delta > 0$ and $S \varepsilon L(B,X)$, there is $T \varepsilon A_\varepsilon$ such that $\|S - T\| < \delta$ and $S - T$ is finite rank.

<u>Proof</u>

Assume $\varepsilon > 0$, $0 < \delta < \frac{1}{2}$ and $S \varepsilon L(B,X)$. Clearly, we can suppose $\sup_{y\varepsilon D}\{\|S(y)\|\} > 0$ and hence $\sup_{y\varepsilon D}\{\|S(y)\|\} = 1$. Now suppose that for every $T \varepsilon L(B,X)$ satisfying $\|S-T\| < \delta$ and $S - T$ finite rank, we have $T \notin A_\varepsilon$. For each $n \varepsilon N$, let V_n be the set of those $x \varepsilon D$ for which there exists $T \varepsilon L(B,X)$ such that

$$\|T(x)\| \geqq \sup_{y\varepsilon D} \|T(y)\| - 4^{-n}\delta^2, \quad \|S - T\| \leqq \delta(1 - 2^{-n}) \text{ and } S - T \text{ has}$$

finite rank.

We claim that if $z \varepsilon V_n$ and $x \varepsilon B$, then

$$d(z, Co(V_{n+1} \setminus U_\varepsilon(x))) < k2^{-n}, \text{ where } k = 2^6\delta.$$

Suppose $d(z, Co(V_{n+1} \setminus U_\varepsilon(x))) \geqq k2^{-n}$. Since $K = (V_{n+1} \setminus U_\varepsilon(x) \cup U_\varepsilon(-x)$ is symmetric, there exists $f \varepsilon B^*$ satisfying $\|f\| = 1$ and $f(z) \geqq \sup|f(K)| + k2^{-n}$.

Because $z \varepsilon V_n$ there is $T \varepsilon L(B,X)$ such that

$$\|T(z)\| \geqq \sup_{x\varepsilon D}\{\|T(x)\|\} - 4^{-n}\delta^2, \quad \|S-T\| \leqq \delta(1 - 2^{-n}) \text{ and}$$

$S - T$ is finite rank. Thus

$$\frac{1}{2} \leq \sup_{y \varepsilon D} \{ \| T(y) \| \} < 2 \quad \text{and}$$

$$\| T(z) \| \geq \frac{1}{4} .$$

Let $\hat{T} \varepsilon L(B, X)$ be the operator given by

$\hat{T}(x) = T(x) + 2^{-n-2} \delta f(x) T(z)$. Then $\| T - \hat{T} \| \leqq 2^{-n+1} \delta$ and hence $\| S - \hat{T} \| \leqq \delta (1 - 2^{-n-1})$. Obviously $S - \hat{T}$ is still a finite rank operator.

By hypothesis $\hat{T} \notin A_{\varepsilon}$ and thus there is some $y \varepsilon D$ with $y \notin U_{\varepsilon}(x) \cup U_{\varepsilon}(-x)$ and $\| \hat{T}(y) \| \geq \sup_{x \varepsilon D} \{ \| \hat{T}(x) \| - 4^{-n-1} \delta^2$. Clearly

$y \varepsilon V_{n+1}$ and thus $y \varepsilon K$.

But $\| T(y) \| + 2^{-n-2} \delta |f(y)| \| T(z) \| \geq \| \hat{T}(z) \| - 4^{-n-1} \delta^2$

implying $(1 + 2^{-n-2} \delta |f(y)|) \| T(z) \| \geq (1 + 2^{-n-2} \delta f(z)) \| T(z) \| - 2 \cdot 4^{-n} \delta^2$.

Therefore, $|f(y)| \geq f(z) - 2^{-n+5} \delta$ which contradicts $f(z) \geq |f(y)| + k 2^{-n}$.

This proves the claim.

From the claim, it follows that the sequence $\{V_n\}_{n=0}^{\infty}$ of nonempty sets in B, $\varepsilon > 0$ and $k > 0$ satisfy the condition of lemma 4.8. Thus $A = \bigcap_{n=0}^{\infty} \overline{\bigcup_{j \geq n} Co(V_j)}$ is nonempty and not dentable.. The fact $A \subseteq D$ yields the final contradiction.

Definition 4.10

Let D be a nonempty, bounded closed and absolutely convex subset of B. Let X be a Banach space and $T \in L(B,X)$. We will say that T is an absolutely strongly exposing operator for the set D if there exists some point $x \in D$ such that every sequence $\{x_n\}$ in D satisfying

$$\sup_{y \in D}\{\|T(y)\|\} = \lim_{n \to \infty} \|T(x_n)\|$$

has a subsequence converging to x or to -x.

Note that using a compactness argument, we observe that $T \in L(B,X)$ is an absolutely strongly exposing operator for the set D if and only if $T \in A_\varepsilon$ for every $\varepsilon > 0$, where A_ε is defined as in lemma 4.9. Obviously such operator T achieves its $\sup_{y \in D}\{\|T(y)\|\}$ on D.

Theorem 4.11

Let D be a nonempty, bounded closed and absolutely convex subset of B. Assume that every nonempty subset of D is dentable. Then for any Banach space X the set A of the absolutely strongly exposing operators $T \in L(B,X)$ for the set D is a dense G_δ -subset of $L(B,X)$. In fact if $S \in L(B,X)$ and $\delta > 0$, there is $T \in A$ such that $\|S - T\| \leqq \delta$ and $S - T$ is a compact operator.

Proof

Clearly D can be taken in the unit ball of B. For each $n = 1,2, \ldots$ consider the subset $A_{1/n}$ of $L(B,X)$, which is open.

Indeed, assume $T \in A_{1/n}$ and let $S(T,\eta) \subseteq U_{1/n}(x) \cup U_{1/n}(-x)$ for some $\eta > 0$ and some $x \in B$. Then, if $R \in L(B,X)$ and

$\|T - R\| < \eta/3$, we have

$S(R, \eta/3) \subseteq S(T,\eta)$ and therefore $R \varepsilon A_{1/n}$. Since $A = \bigcap_n A_{1/n}$, it follows from lemma 4.9 that A is a dense G_δ set in $L(B,X)$.

Now assume $S \varepsilon L(B,X)$ and $\delta > 0$. Let ϕ be the set of compact operators C in $L(B,X)$ such that $\|C\| \leq \delta$. Then $S + \phi$ is closed in $L(B,X)$ and again from lemma 4.9 we obtain that $(S+\phi) \bigcap A_{1/n}$ is dense in $S + \phi$ for each $n = 1,2, \ldots$. Therefore A intersects $S + \phi$ and every operator T in the intersection verifies the required properties.

Corollary 4.12

Let D be a nonempty, bounded closed and absolutely convex subset of B. Assume that every nonempty subset of D is dentable. Then for any Banach space X the set of those operators $T \varepsilon L(B,X)$ which attains there maximum norm $\sup_{y \varepsilon D}\{\|T(y)\|\}$ in D is dense in $L(B,X)$. Hence D has the Bishop-Phelps property.

Finally corollary 4.12 gives.

Theorem 4.13

If a Banach space B has RNP then B has BPP.

We end this section by the following :

Lemma 4.14

Let $K \subseteq B$ be closed, bounded convex set with RNP. Then $L = \{t(x,-1) : 0 \leq t \leq 1; x \varepsilon K\}$ has RNP.

Proof

Suppose that $(X, \sum,\mu)$ is a probability space and $m : \sum \to B \times R$ is a vector measure with $A\,R(m) \subseteq L$. Write m in the form (m_1, m_2)

where $m_1 : \sum \to B$ and $m_2 : \sum \to R$ are measures. Since $AR(m) \subseteq L$ and so $AR(m_2) \subseteq [0,1]$ there is an $f \in L^1_{[0,1]}(\mu)$ with $\int_A f d\mu = m_2(A)$ for $A \in \sum$ by the classical RNT. Now, suppose $A \in \sum$ and $m_2(A) \neq 0$. Then

$$(*)\ \frac{m(A)}{\mu(A)} = \left[\frac{m_1(A)}{\mu(A)}, \frac{m_2(A)}{\mu(A)}\right] = \frac{-m_2(A)}{\mu(A)} \left[\frac{m_1(A)}{-m_2(A)}, -1\right].$$

Note that $\dfrac{-m_2(A)}{\mu(A)} \in [0,1]$ and $\dfrac{m(A)}{\mu(A)} \in AR(m) \subseteq L$.

But for each $z \in L$ with $z \neq 0$ there is a unique choice of $p \in [0,1]$ and $x \in K$ such that $z = p(x,-1)$.

Hence from (*) we conclude $\dfrac{m_1(A)}{-m_2(A)} \in K$. Thus

$\left\{ \dfrac{m_1(A)}{-m_2(A)} : A \in \sum, -m_2(A) > 0 \right\} \subseteq K$ and since K has the RNP there is a $g \in L^1_K(-m_2)$ with $\int_A g\, d(-m_2) = m_1(A)$ for each $A \in \sum$, consequently

$$\int_A -fg\, d\mu = m_1(A) \text{ for each } A \in \sum \text{ hence}$$

$\dfrac{dm}{d\mu}$ exists and is $(-fg, f)$. The proof is complete.

2. This section contains a generalisation of the Bourgin definition of the BPP 4.2 to a quasi-complete ℓ.c.s's with BM Property. The heart of this section is the generalisation of theorem 4.7 to a quasi-complete ℓ.c.s. with BM property.

Notation 4.15

Let E denote a quasi-complete ℓ.c.s. with BM property and $\{p_\alpha\}_{\alpha\in\Lambda}$ is a family of semi-norms that makes E a quasi-complete ℓ.c.s. Let K be closed bounded subset of E. For any Banach space Y let $L(E,Y)$ denote the set of bounded linear operators on E into Y with

$$q_\alpha(T) = \sup\{\| T(x)\| : p_\alpha(x) \leq 1\}.$$

To prove now q_α defined above is a semi-norm for each $\alpha \in \Lambda$ that makes $L(E,Y)$ a ℓ.c.s.

(i) $q_\alpha(T) = 0$ implies $\sup\{\| T(x)\| : p_\alpha(x) \leq 1\} = 0$ and this gives $\| T(x)\| = 0$ and this implies that $T(x) = 0$ for $p_\alpha(x) \leq 1$ for all α and for any $x \in E$ so $T = 0$.

(ii) $q_\alpha(\lambda T) = \sup\{\| (\lambda T)(x)\| : p_\alpha(x) \leq 1\} = \sup\{|\lambda| \, \|T(x)\| : p_\alpha(x) \leq 1\}$

$$= |\lambda| \sup\{\| T(x)\| : p_\alpha(x) \leq 1\} = |\lambda| \, q_\alpha(T).$$

(iii) $q_\alpha(T_1+T_2) = \sup\{\| (T_1 + T_2)(x)\| : p_\alpha(x) \leq 1\}$

$$\leq \sup\{\| T_1(x)\| + \| T_2(x)\| : p_\alpha(x) \leq 1\}$$

$$\leq \sup\{\| T_1(x)\| : p_\alpha(x) \leq 1\} + \sup\{\|T_2(x)\| : p_\alpha(x) \leq 1\}$$

$$= q_\alpha(T_1) + q_\alpha(T_2) \quad \text{for each } T_1, T_2 \in L(E,Y).$$

So q_α is a semi-norm for each $\alpha \in \Lambda$ and this family of semi-norms makes $L(E,Y)$ a locally convex space.

Definition 4.16

Let E and K as in notation 4.15 then K has BPP if whenever Y is a Banach space and $T \in L(E,Y)$ then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in $L(E,Y)$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in K such that for any n.

(i) $q_\alpha(T_n) < \frac{1}{n}$ for each $\alpha \in \Lambda$ and

(ii) $\|(T+T_n)(x_n)\| = \sup_y \{\|(T+T_n)(y)\| : y \in K\}$.

The next theorem is a generalization of the proposition 4.6 in section 1 into a quasi-complete ℓ.c.s with BM property. Before we give this theorem we need some notation.

Notation 4.17

Let E be a quasi-complete ℓ.c.s. with BM property and let C be a closed bounded convex subset of E. Let $M = \overline{Co}(C \cup -C)$ and let $E_M = \bigcup_{n=1}^{\infty} n\,M$. Let V_n be a closed absolutely convex 0-nhd in E. Since M is bounded, for every n there exist $a_n \geq 1$ such that $M \subseteq a_n V_n$. Let p_n be the gauge functional of V_n. For every $x \in E_M$ define $N(x) = \sum_{n=1}^{\infty} \frac{1}{a_n 2^n} P_n(x)$. This N is a norm on E_M see Saab [43].

Let $\hat{E}_M$ be the completion of (E_M, N)

Theorem 4.18

Let C be a separable non-empty closed bounded convex subset of E. If C has the BPP then C is dentable.

Proof

Let $M = \overline{Co}(C \cup -C)$ and $E_M = \bigcup_{n=1}^{\infty} nM$ and let N be a norm on $\hat{E}_M$ as in the notation 4.17. We have $C \subseteq \hat{E}_M$ and C is a closed bounded, convex separable subset of $\hat{E}_M$. So by proposition 4.7 C is dentable in $\hat{E}_M$ therefore C is dentable in E by corollary 2.2(i) of Saab [43].

Theorem 4.19

Let E be a quasi-complete $\ell.c.s.$ with BM property, and let K be a closed, bounded, convex subset of E. If every closed, bounded, convex separable subset of K has the RNP then K has the RNP in E.

Proof

Let $M = \overline{Co}(K \cup -K)$ and E_M, $\hat{E}_M$, N be defined as in Notation 4.17. $K \subseteq \hat{E}_M$ be a closed bounded convex subset of $\hat{E}_M$ and let C be closed, bounded convex separable subset of K and C has RNP in $\hat{E}_M$, so K has RNP in E by corollary 2.4 of Saab [43].

We end this section by the following theorem which is a generalization of theorem 4.7 in quasi-complete $\ell.c.s$ with BM property.

Theorem 4.20

Let K be closed, bounded, convex subset of E. If each separable closed, bounded convex subset of K has the BPP then K has the RNP.

Proof

Let $C \subseteq K$ be a separable, closed, bounded and convex and suppose that C has the BPP. Then by Theorem 4.18 C is dentable and so C has the RNP by virtue of Theorem 2.5 of Saab [43], so K has the RNP by Theorem 4.19.

3. This section contains a new definition of BPP on a class of quasi-complete ℓ.c.s. with BM property. This is given to help us prove the equivalence of the BPP and the RNP. We state in this section the definition of the BPP which is given by Egghe [22]. We also prove that the extension of the Bourgin definition of the BPP gives the Egghe definition and the latter gives our definition of the definition of the BPP. We end this section by proving that the RNP gives the BPP in the sense of our definition in a class of quasi-complete ℓ.c.s. with BM property.

Definition 4.21 (Egghe definition of the BPP*.)

Let E be a ℓ.c.s then E has the BPP* if for every Banach space Y and every $S \in L(E,Y)$ and for every closed, bounded, absolutely convex subset $D \subseteq E$ there exists a sequence $\{S_n\}_{n=1}^{\infty}$ in $L(E,Y)$, such that $S_n \to S$ for $n \to \infty$ uniformly on the bounded subsets of E, and such that for any $n \in N$ there exists $x_n \in D$ such that

$$\| S_n(x_n) \| = \sup_{y \in D} \| S_n(y) \| .$$

Definition 4.22

Let E be a quasi-complete ℓ.c.s. with BM property and K be a closed, bounded, convex subset of E. Let F be the subspace generated by K. Then K is said to have BPP** if whenever Y is a Banach space and $r \in L(F,Y)$ then there exists sequences $\{x_n\}_{n=1}^{\infty}$ in K and $r_n \in L(F,Y)$ such that for each n.

(i) $q_\alpha\big|_F (r_n) \leq \frac{1}{n}$ for each $\alpha \in \Lambda$ and

(ii) $\|(r_n + r)(x_n)\| = \sup_{y \in K}\{\|(r + r_n)(y)\|\}$.

If every closed, bounded convex subset of E has the BPP^{**} then E has BPP^{**}.

The next Theorem gives the relation between BPP, BPP^{*}, and BPP^{**}.

Theorem 4.23

Let E be quasi-complete ℓ.c.s. with BM property then the following implications are true 4.16 $\Longrightarrow$ 4.21 $\Longrightarrow$ 4.22

Proof

Let E be quasi-complete ℓ.c.s with BM property. Let $C \subseteq E$ be closed, bounded, absolutely convex and having the BPP. We need to prove now that for every Banach space Y and $S \in L(E,Y)$ and for every $D \subseteq E$ closed, bounded absolutely convex subset of E there exists a sequence $\{S_n\}_{n=1}^{\infty}$ in $L(E,Y)$ such that $S_n \to S$ for $n \to \infty$ uniformly on the bounded sets of E, and there exists $\{x_n\}_{n=1}^{\infty}$ in D such that

$$\| S_n(x_n) \| = \sup_{y \in D} \{\| S_n(y) \|\}.$$

Since D is closed bounded absolutely convex then D has BPP i.e. for every Banach space Y and $T \in L(E,Y)$ there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in $L(E,Y)$ and a sequence $\{x_n\}$ in D such that for each n

(i) $q_\alpha(T_n) \leq \frac{1}{n}$ for each $\alpha \in \Lambda$ and

(ii) $\| (T_n + T)(x_n) \| = \sup_{y \in D}\{\| (T_n + T)(y) \|\}$.

Take $S = T$ and $S_n = T_n + T$ we get

$q_\alpha(S_n - S) = q_\alpha(T_n) \leq \frac{1}{n}$ for every n and for each $\alpha \in \Lambda$. So

$q_\alpha(S_n - S) \to 0$ when $n \to \infty$ uniformly on bounded sets of E.

$$\| S_n(x_n) \| = \| (T + T_n)(x_n) = \sup_{y \in D}\{\| (T + T_n)(y) \|\} = \sup_{y \in D}\{S_n(y)\| \}.$$

So E has BPP $*$.

<u>4.16 ==> 4.22</u>

Suppose that E has BPP^* and let D be closed, bounded absolutely convex subset of E and F the subspace generated by D.

For each Banach space Y and for each $T \in L(E, Y)$ there exists a sequence $\{T_n\}_{n=1}^\infty$ in $L(E, Y)$ such that $T_n \to T$ uniformly on the bounded subsets of E and such that there exists a sequence $\{x_n\}_{n=1}^\infty$ in D such that

$$\| T_n(x_n) \| = \sup_{y \in D}\{ \| T_n(y) \|\} .$$

Take $S_n = T_n\big|_F$ and $S = T\big|_F$

then $S_n \to S$ uniformly on bounded sets of E and

$$\| S_n(x_n) \| = \sup_{y \in D} \{ \|S_n(y) \|\} .$$

Now take $r_n = S_n - S$ and $r = S$ we get

(i) $q_\alpha\big|_F (r_n) = q_\alpha\big|_F (S_n - S) \leq \frac{1}{n}$ for each n and $\alpha \in \Lambda$.

(ii) $\| (r_n + r)(x_n) \| = \| S_n(x_n) \| = \sup_{y \in D} \{\| S_n(y) \| \}$

$$= \sup_{y \in D} \{ \|(r_n + r)(y) \| \} .$$

So D has BPP ** and since D has the BPP** and since D is arbitrary so E has BPP**.

We end this section by the following theorem which is the heart of this section and is the converse of theorem 4.20.

Theorem 4.25

Let E be quasi-complete ℓ.c.s. with BM property and let K be closed, bounded convex subset of E and assume that K has the RNP. Then K has BPP **.

Proof

Let $M = \overline{Co}(K \cup -K)$ and $E_M = \bigcup_{n=1}^{\infty} nM$ and $(\hat{E}_M, N)$ be the completion of E_M with norm N (see notation 4.17). Since $K \subseteq M$ so $K \subseteq E_M \subseteq \hat{E}_M$ and K is closed bounded, convex and has the RNP so by theorem 4.13 for any Banach space Y and $S \in L(\hat{E}_M, Y)$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in K such that $N(S_n) \leq \frac{1}{n}$ for each n and

$$\|(S_n + S)(x_n)\| = \sup_{y \in K} \|(S_n + S)(y)\|\} .$$

Now let F be the subspace generated by K and let $T \in L(F,Y)$ be given. Clearly, $F \subseteq E_M$ since $K \subseteq E_M$. Define $\{T_n\}_{n=1}^{\infty}$ by $T_n(x) = S_n(x)$ if $x \in F$. So

(i) $(q_\alpha|_F)(T_n) \leq N(S_n) \leq \frac{1}{n}$ for each n and α,

(ii) $\|(T_n + T)(x_n)\| = \|(S_n + S)(x_n)\| = \sup_{y \in K}\{\|(S_n + S)(y)\|$.

$$= \sup_{y \in K} \|(T + T_n)(y)\|$$

so K has BPP ** in F and so K has BPP ** in E by definition 4.22.

4. This section contains a generalization of the result of Saab[43] to the cross product of two quasi-complete ℓ.c.s's with BM property. We use this generalization to prove lemma 4.4 in quasi-complete ℓ.c.s with BM property. We end this section by proving that every barrelled space has BM property and we give an example to show that this is not in general true.

We first prove that the cross product of two quasi-complete ℓ.c.s with BM property is a quasi-complete ℓ.c.s with BM property.

Lemma 4.26

Let (E,τ) and (F,ρ) be two quasi-complete ℓ.c.s's with BM property then $(E \times F, \tau \times \rho)$ is a quasi-complete ℓ.c.s with the same property.

Proof

Let $\{p_\alpha\}_{\alpha\varepsilon\Lambda}$ and $\{q_\beta\}_{\beta\varepsilon\gamma}$ be a two family of semi norms generated by the topologies τ and ρ respectively. Define a new family of semi-norms on $E \times F$ as follows: $r(x) = \max(p_\alpha(x_1), q_\alpha(x_2))$ where $x = (x_1, x_2)$ so $E \times F$ is a quasi-complete ℓ.c.s. (product of any family of quasi-complete ℓ.c.s. is quasi-complete ℓ.c.s. see Shaefer [44] page 27).

Let K be closed, bounded, subset of $E \times F$ therefore, there is a bounded closed subset $C \subseteq E$ and $S \subseteq F$ such that $K \subseteq C \times S$. Since C and S are complete and metrizable then so is $C \times S$, therefore K is complete and metrizable i.e. $E \times F$ has BM property

Let $E \times F$ be a quasi-complete ℓ.c.s. with BM property and let K be closed bounded convex subset of $E \times F$, Let $M = \overline{Co}(K \cup -K)$ and $(E \times F)_M = \bigcup_{n=1}^{\infty} nM$. So we can state the following.

Theorem 4.27

There exists a norm N on $(E \times F)_M$ such that the topology induced by $((E\times F)_M, N)$ on M coincides with the topology induced by $(E\times F,\tau\times\rho)$ of M.

Proof

Let $K \subseteq E\times F$ be closed, bounded and convex subset of $E\times F$, then there exists closed bounded convex subsets $J \subseteq E$ and $H \subseteq F$ such that $K \subseteq J\times H$. Now let $L = \overline{\mathrm{Co}}(J\cup -J)$ and $P = \overline{\mathrm{Co}}(H\cup -H)$, and $E_L = \bigcup_{n=1}^{\infty} n\,L$ and $F_P = \bigcup_{n=1}^{\infty} n\,P$, then by theorem 2.1 of Saab [43] there exists a norm N_ℓ on E_L such that the induced topology by (E_L, N_ℓ) on L coincides with the topology induced by (E,τ) on L, and there exists a norm N_p on F_P such that the induced topology by (F_P, N_p) on P coincides with the topology induced by (F, p) on P. Define a new norm N_t on $E_L\times F_P$ by

$$N_t(x) = \max\,(N_\ell(x_1)\ ,\ N_p(x_2)) \quad \text{where} \quad x = (x_1, x_2).$$

Clearly, N_t is a norm and the topology induced by N_t on $L\times P$ coincides with the topology induced by $(E\times F,\tau\times\rho)$ on $L\times P$, by construction.

Now $M \subseteq L\times P$ so $(E\times F)_M \subseteq E_L\times F_P$. Define N on $(E\times F)_M$ as follows:

$M(x) = N_t\big|_{(E\times F)_M}(x)$ so from this definition we get the result. Now since L is complete in E_L and P is complete in F_P so let $\hat{E}_L$, $\hat{F}_P$ be the completion of E_L and F_P respectively. So

$$\widehat{(E\times F)}_M \subseteq \widehat{E_L\times F_P} \subseteq \hat{E}_L\times\hat{F}_P \quad \text{for if,}$$

$(x,y)\in(E\times F)_M$ so there exists a net (x_α, y_α) such that $(x_\alpha, y_\alpha) \to (x,y)$ since $x_\alpha\in E_L$, $y_\alpha\in F_P$ so by definition of the product topology

$$x_\alpha \to x \text{ and } y_\alpha \to y \text{ so } \widehat{(F_L \times F)}_M \subseteq \hat{E}_L \times \hat{F}_P.$$

The next two corollaries are direct consequence of the above theorem and corollary 2.2 and theorem 2.3 of Saab [43].

Corollary 4.28

Let K and M be as in theorem 4.25, then

(i) the set K is dentable (σ-dentable) in $(E \times F, \tau \times \rho)$ if and only if K is dentable (σ-dentable in $((E \times F)_M, N)$

(ii) a point $x \in K$ is a denting point in $(E \times F, \tau \times \rho)$ if and only if x is a denting point in $((E \times F)_M, N)$.

Corollary 4.29

Let $(X, \sum, \mu)$ be a probability space and let K and M be as theorem 4.20 then a function $f\colon X \to K$ is μ-integrable in $(E \times F, \tau \times \rho)$ if and only if f is μ-integrable in $((E \times F)_M, N)$ in this case $\int_X f\,d\mu$ in $(E \times F)_M$ is the same as $\int_X f\,d\mu$ in $((E \times F), \tau \times \rho))$ for every $X \in \sum$.

The next corollary is direct from corollary 4.27 and theorem 4.26.

Corollary 4.30

If K and M as in theorem 4.20 then K has the RNP in $(E \times F, \tau \times \rho)$ if and only if K has the RNP in $((E \times F)_M, N)$.

Corollary 4.31

If K be a closed, bounded convex subset of $E \times F$ then the following are equivalent

(i) the set K has the RNP.

(ii) the set K is subset dentable.

(iii) the set K is subset σ-dentable.

The proof of the above corollary is direct from corollary 4.29 and 4.27 and the result of Saab [43].

Theorem 4.32

Let K be a closed, bounded, convex subset of $E \times F$ then the following assertions are equivalent.

(i) every closed, convex subset of K is the closed convex hull of its denting points.

(ii) The set K has the RNP.

Proof

(ii) $\to$ (i) Let $M = \overline{Co}(K \cup -K)$ and consider $M \subseteq (E \times F)_M$, Let C be a closed bounded convex subset of K then C has RNP in $E \times F$ and therefore C has RNP in $(E \times F)_M$. So by theorem 2.6 of Saab [43] C is the closed convex hull of its strongly exposed points in $(\widehat{E_1 \times F})_M$ in particular of its denting points in $(\widehat{E \times F})_M$ so an appeal to theorem 4.21 and corollary 4.27 finishes the proof. The other implication is immediate from the definition of the denting point and the corollary 4.25.

The next lemma is a generalization of the lemma 4.14 in quasi-complete ℓ.c.s. with BM property.

Lemma 4.33

Let $K \subseteq E$ be a closed, bounded and convex subset of E with the RNP then the set $L = \{(tx, -t) : 0 \leqq t \leqq 1,\ x \in K\}$ has the RNP.

Proof

Let $M = \overline{Co}(K \cup -K)$ and $E_M = \bigcup_{n=1}^{\infty} nM$ and $\hat{E}_M$ be the completion of $\hat{E}_M$ then $K \subsetneq \hat{E}_M$ is closed, bounded and convex with the RNP then L has RNP in $\hat{E}_M \times [-1,0]$ by lemma 4.9 so $L \subseteq \hat{E}_M \times [-1,0]$ and so by theorem 4.26 and corollary 4.27 we get the result.

We end this section by proving that every closed bounded subset of a barreled space is metrizable.

Theorem 4.34

Let E be Hausdorff barreled space and let C be a closed bounded subset of E. Then C is metrizable.

Proof

Take $a\overline{Co}(C)$; we have,

(i) $a\overline{Co}(C)$ is bounded by lemma 1 page 44 of Robertson and Robertson [41].

(ii) $a\overline{Co}(C)$ is Hausdorff since E is Hausdorff.

Let $L(a\overline{Co}(C)) = L$ be the subspace generated by $a\overline{Co}(C)$. So L is a locally convex space which is Hausdorff and locally bounded because (i) holds and $a\overline{Co}(C)$ is an absorbent, balanced absolutely convex subset of L so $a\overline{Co}(C)$ is a nhd of 0 since E is barrelled. So L is locally bounded, and so L is metrizable (every locally bounded Hausdorff T.V.S. is metrizable.)

Since $C \subseteq L$ so C is a metrizable [every subset of a metrizable space is metrizable].

Corollary 4.35

Let E be a Hausdorff barreled space and let C be a bounded subset of E. Then C is metrizable.

Proof

Since C is bounded so is $\bar{C}$. Then by theorem 4.33 $\bar{C}$ is metrizable. So is C.

Note that the proof of theorem 4.28 does not work with a general ℓ.c.s. because $\overline{aCo(C)}$ is not always nhd of 0 in a general space.

Example 4.36

Let E be the vector space over R of all continuous real-valued functions f on $[0,1]$ that vanish in a neighbourhood (depending on f) of $t=0$, under the uniform topology. Let $D = \{f : n|f(n^{-1})| \leq 1, n \in N\}$; we have to show that D is a barrel in E but not a nhd of 0.

(i) D is convex: let $f, g \in D$ $0 \leq \lambda \leq 1$, then

$$n\cdot|(\lambda f + (1-\lambda)\, g)(n^{-1})| \leq n\, \lambda|f(n^{-1})| + n(1-\lambda)\, g(n^{-1})| < \lambda + 1 - \lambda = 1$$

so D is convex.

(ii) D is balanced: Let $f \in D$ and $|\lambda| \leq 1$ so

$$n\cdot|\lambda(f(n^{-1})| = n\,|\lambda| \cdot |\, f(n^{-1})| \leq |\lambda| \leq 1 \quad \text{so } D \text{ is balanced.}$$

(iii) D is absorbent: i.e. to prove for all $f \in E$ there is λ such that $f \in \mu D$ for all μ with $|\mu| \geq \lambda$. Let $f \in E$ to prove $\frac{1}{\mu} f \in D$

for all $|\mu| \geqq \lambda$. Let $0 < \delta < 1$ be such that $f = 0$ in $[0,\delta]$ and so $\frac{1}{n} < \delta$ for $n > n_o$. We want

$$\frac{1}{\mu} |f(n^{-1})| \leq \frac{1}{n} \qquad n = 1,2,3, \ldots, n_o$$

so set $$\lambda = \max_{1 \leq n \leq n_o} n|f(n^{-1})|$$

when $|\mu| > \lambda$ then $f \in \mu D$ i.e. $\frac{1}{\mu} f \in D$.

So by (i), (ii), and (iii), D is a barrel in E.

To prove now D is not a nhd of 0. Let $\varepsilon > 0$ and define f_n by

$$f_n(t) = \begin{cases} \frac{3}{4}\varepsilon & \text{if } t \geq \frac{1}{n} \\ \frac{3}{2}\varepsilon nt - \frac{3}{4}\varepsilon & \text{if } \frac{1}{2n} \leq t \leq \frac{1}{n} \\ 0 & \text{if } t < \frac{1}{2n} \end{cases}$$

f_n is continuous and $f_n \in E$,

$$\| f_n \|_\infty < \varepsilon$$

$$m | f_n(\tfrac{1}{m}) | = m \left| \tfrac{3}{4}\varepsilon \right| \quad \text{if } \frac{1}{m} > \frac{1}{n}$$

$$= \frac{3}{4} m \varepsilon > 1,$$

So choosing n,m with $n > m > 4/3\varepsilon$ we have $f_n \notin D$ as required.

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